International Journal of Science & Technology
Volume 1, No 1, 11-14, 2006

A New Type Of Difference Sequence Spaces

Binod CHANDRA TRIPATHY[1] **and AYHAN ESI**[2]
[1]Mathematical Sciences Division; Institute of Advanced Study in Science and Technology; Paschim Boragaon; GARCHUK; GUWAHATI-781 035, INDIA
[2]Inonu University, Science and Art Faculty in Adiyaman, 02040, Adıyaman, TURKEY
tripathybc@yahoo.com

(Received: 17.08.2005; Accepted: 12.12.2005)

Abstract: In this paper we introduce the notion of the difference operator $\Delta_m x_k$ for a fixed $m \in N$. We define the sequence spaces $\ell_\infty(\Delta_m), c(\Delta_m)$ *and* $c_o(\Delta_m)(m \in N)$ and study some topological properties of these spaces. We obtain some inclusion relations involving these sequence spaces.

2000 AMS . 40 A 05; 40 C 05; 46 A 45.

Key words: Difference sequence space, Solid space, Symmetric space, Completeness

Fark Dizi Uzaylarının Yeni Bir Şekli

Özet: Bu çalışmada sabit bir $m \in N$ sayısı için $\Delta_m x_k$ fark operatörü yardımıyla $\ell_\infty(\Delta_m), c(\Delta_m)$ *ve* $c_o(\Delta_m)$ dizi uzayları tanımlanıp, bu uzaylar için bazı topolojik özellikler çalışılmış ve bu uzaylara ait bazı kapsam bağıntıları verilmiştir.

Anahtar Kelimeler: Fark dizi uzayı, Solid uzay, Simetrik uzay, Tamlık

1. Introduction

Throughout the paper w, ℓ_∞, c, and c_0 denote the spaces of *all*, *bounded*, *convergent*, and *null* sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\| x \|_\infty = \sup_k |x_k| .$$

The zero sequence is denoted by $\theta = (0,0,0, . .)$.

Kızmaz [3] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ as follows:

$$Z (\Delta) = \{ x = (x_k) : (\Delta x_k)\ \Delta\ Z \},$$

for $Z = \ell_\infty, c$ and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in N$.

The above spaces are Banach spaces, normed by

$$\| x \|_\Delta = | x_1| + \sup_k \| \Delta x_k \| .$$

The idea of Kizmaz [3] was applied to introduce different type of difference sequence spaces and study their different properties by Tripathy ([6], [7]), Et and Esi [8] and many others.

2. Definitions and Preliminaries

A sequence so ace E said to be *solid* (or *normal*) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $| \alpha_k | \leq 1$ for all $n \in N$.

A sequence space E is said to be *monotone* if it contains the canonical preimages of all its step spaces.

A sequence space E is said to be *convergence free* if $(y_k) \in E$ whenever $(x_k) \in E$ and $y_k = 0$ whenever $x_k = 0$.

A sequence space E is said to be a *sequence algebra* if $(x_k.y_k) \in E$ whenever $(x_k) \in E$ and $(y_k) \in E$.

A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$, where $\pi(k)$ is a permutation on N .

For $r > 0$, a nonempty subset V of a linear space is said to be absolutely *r-convex* if $x, y \in V$

and $|\lambda|^r + |\mu|^r \leq 1$ together imply that $\lambda x + \mu y \in V$. A linear topological space X is said to be r-convex if every neighborhood of $\theta \in X$ contains as absolutely r-convex neighborhood of $\theta \in X$ (see for instance Maddox and Roles [5]).

Let $m \in N$ be fixed, then we introduce the following new type of difference sequence spaces

$$Z(\Delta_m) = \{x = (x_k) \in w: \quad \Delta_m x \in Z\},$$

for $Z = \ell_\infty, c$ and c_0, where

$\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$, for all $k \in N$.

For $m = 1$, $\ell_\infty(\Delta_m) = \ell_\infty(\Delta)$, $c(\Delta_m) = c(\Delta)$ and $c_0(\Delta_m) = c_0(\Delta)$. Hence the introduced notion generalizes the notion of difference sequences studied by Kizmaz [3].

3. Main Results

In this section we establish the results of this article. The proof of the following result is a routine verification.

Proposition 1. *The classes of sequences $\ell_\infty(\Delta_m), c(\Delta_m)$ and $c_o(\Delta_m)$ are normed linear spaces, normed by*

$$\|x\|_{\Delta_m} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k| \quad \text{-------} \ (1)$$

Proof. Let α, β be scalars and $(x_k), (y_k) \in \ell_\infty(\Delta_m)$. Then

$$\sup_k |\Delta_m x_k| < \infty \quad \text{and} \quad \sup_k |\Delta_m y_k| < \infty$$

--------- (2)

Hence

$$\sup_k |\Delta_m(\alpha x_k + \beta y_k)| \leq |\alpha| \sup_k |\Delta_m x_k| + |\beta| \sup_k |\Delta_m y_k| < \infty, \text{ by (2)}.$$

Hence $\ell_\infty(\Delta_m)$ is a linear space. Similarly it can be shown that $c(\Delta_m)$ and $c_0(\Delta_m)$ are linear spaces.

Next for $x = \theta$, we have $\|\theta\|_{\Delta_m} = 0$. Conversely, let $\|x\|_{\Delta_m} = 0$.

Then

$$\|x\|_{\Delta_m} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k| = 0.$$

$\Rightarrow x_r = 0$ for $r = 1, 2, ---, m$ and $|\Delta_m x_k| = 0$, for all $k \in N$.

Consider $k = 1$ *i.e.* $|\Delta_m x_1| = 0 \Rightarrow |x_1 - x_{1+m}| =$

$\Rightarrow x_{m+1} = 0$, since $x_m = 0$.

Proceeding in this way we have $x_k = 0$, for all $k \in N$.

$$\|x + y\|_{\Delta_m} = \sum_{r=1}^{m} |x_r + y_r| + \sup_k |\Delta_m(x_k + y_k)|$$

$$\leq \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k| + \sum_{r=1}^{m} |y_r| + \sup_k |\Delta_m y_k|$$

$$= \|x\|_{\Delta_m} + \|y\|_{\Delta_m}.$$

Finally

$$\|\lambda x\|_{\Delta_m} = \sum_{r=1}^{m} |\lambda x_r| + \sup_k |\Delta_m(\lambda x_k)|$$

$= |\lambda| \, \|x\|_{\Delta_m}$.

Hence $\|.\|_{\Delta_m}$ is a norm on the spaces $\ell_\infty(\Delta_m), c(\Delta_m)$ and $c_o(\Delta_m)$.

This completes the proof.

Proposition 2. (*i*) $c_o(\Delta_m) \subset c(\Delta_m) \subset \ell_\infty(\Delta_m)$ *and the inclusions are proper.*

(*ii*) $Z(\Delta) \subset Z(\Delta_m)$, *for* $Z = c, c_0, \ell_\infty$ *and the inclusions are strict.*

Proof. Trivial.

Theorem 3. *The sequence spaces $\ell_\infty(\Delta_m), c(\Delta_m)$ and $c_o(\Delta_m)$ are Banach spaces under* the *norm* (1) .

Proof. Let (x^n) be a Cauchy sequence in $\ell_\infty(\Delta_m)$, where $(x^n) = (x_i^n) = (x_1^n, x_2^n, x_3^n, \ldots) \in \ell_\infty(\Delta_m)$ for each $n \in N$.

Then

$$\|x^n - x^i\|_{\Delta_m} = \sum_{r=1}^{m} |x_r^n - x_r^i| + \sup_k |\Delta_m x_k^n - \Delta_m x_k^i| \to 0$$

, as $n, i \to \infty$. Hence for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$\left\|x^n - x^i\right\|_{\Delta_m} = \sum_{r=1}^{m}\left|x_r^n - x_r^i\right| + \sup_k |\Delta_m x_k^n - \Delta_m x_k^i| < \varepsilon$$

, for all $n, i \ge n_0$. - - - - (3)

Hence $\left|x_k^n - x_k^i\right| < \varepsilon$ for all $n, i \ge n_0$ and $k = 1, 2, \ldots, m$.

$\Rightarrow \left(x_k^i\right)$ is a Cauchy sequence in *C* for $k = 1, 2, \ldots, m$.

$\Rightarrow \left(x_k^i\right)$ is a convergent in *C* for $k = 1, 2, \ldots, m$.

Let $\lim_{i\to\infty} x_k^i = x_k$, say for $k = 1, 2, \ldots, m$.

From (3) we have $| \Delta_m x_k^n - \Delta_m x_k^i | < \varepsilon$, for all $n, i \ge n_0$ and all $k \in N$. Hence $\left(\Delta_m x_k^i\right)$ is a Cauchy sequence in C for all $k \in N$. Thus $\left(\Delta_m x_k^i\right)$ is convergent in *C*, let $\lim_{i\to\infty} \Delta_m x_k^i = y_k$, say for each $k \in N$. Since $\lim_{i\to\infty} x_k^i = x_k$, exists for $k = 1, 2, \ldots, m$, so we have $\lim_{i\to\infty} x_k^i = x_k$, exists for each $k \in N$.

We have

$$\lim_{j\to\infty} \sum_{r=1}^{m} | x_r^i - x_r^j | = \sum_{r=1}^{m} | x_r^i - x_r | < \varepsilon ,$$

for all $i \ge n_0$, and

$$\lim_{j\to\infty} | x_k^i - x_k^j - (x_{k+m}^i - x_{k+m}^j) |$$

, for all $k \in N$ and

$$= | x_k^i - x_k - (x_{k+m}^i - x_{k+m}) | < \varepsilon$$

$i \ge n_0$.

Hence for all $i \ge n_0$, we have

$$\sup_k | \Delta x_k^i - \Delta x_k | < \varepsilon .$$

Thus

$$\sum_{r=1}^{m} | x_r^i - x_r | + \sup_k | \Delta x_k^i - \Delta x_k | < 2\varepsilon ,$$

$\Rightarrow (x^i - x) \in \ell_\infty(\Delta_m)$, for all $i \ge n_0$.

Thus $x = x^i - (x^i - x) \in \ell_\infty\left(\Delta_m\right)$, for all $i \ge n_0$, since $\ell_\infty\left(\Delta_m\right)$ is a linear space.

Hence $\ell_\infty\left(\Delta_m\right)$ is complete.

Similarly it can be shown that the spaces $c\left(\Delta_m\right)$ and $c_o\left(\Delta_m\right)$ are also complete.

The following result is a consequence of the above result and the definition of *BK*-space.

Proposition 4. *The spaces* $\ell_\infty\left(\Delta_m\right), c\left(\Delta_m\right) and\ c_o\left(\Delta_m\right)$ *are BK-spaces.*

Since the inclusions $c\left(\Delta_m\right) \subset \ell_\infty\left(\Delta_m\right)$ and $c_o\left(\Delta_m\right) \subset \ell_\infty\left(\Delta_m\right)$ are proper, the following result follows from Theorem 3.

Proposition 5. *The spaces* $c\left(\Delta_m\right)$ *and* $c_o\left(\Delta_m\right)$ *are nowhere dense subsets of* $\ell_\infty\left(\Delta_m\right)$.

Theorem 6. *The spaces* $\ell_\infty\left(\Delta_m\right), c\left(\Delta_m\right) and\ c_o\left(\Delta_m\right)$ *are not solid spaces.*

Proof. The proof follows from the following examples.

Example 1. Let $x_k = k$ for all $k \in N$. Consider the sequence of scalars (α_k) defined by $\alpha_k = im + 1$, for $i = 0, 1, 2, \ldots$ and $\alpha_k = 0$, otherwise. Then $(x_k) \in c\left(\Delta_m\right) \subset \ell_\infty\left(\Delta_m\right)$, but $(\alpha_k x_k) \notin \ell_\infty\left(\Delta_m\right)$. Hence the spaces $c\left(\Delta_m\right)$ and $\ell_\infty\left(\Delta_m\right)$ are not solid.

For the case $c_o\left(\Delta_m\right)$, consider the sequence $x_k = 1$ for all $k \in N$ and the sequence (α_k) defined as above. Then $(x_k) \in c_o\left(\Delta_m\right)$, but $(\alpha_k x_k) \notin c_o\left(\Delta_m\right)$. Hence $c_o\left(\Delta_m\right)$ is not solid.

Theorem 7. (*i*) *The space* $c_o\left(\Delta\right)$ *is symmetric.*
(*ii*) *The spaces* $\ell_\infty\left(\Delta_m\right), c\left(\Delta_m\right) and\ c_o\left(\Delta_m\right)$ (for $m > 1$) *are not symmetric spaces.*

Proof. (*i*) The first part is known. For the second part, consider the following example.

Example 2. Let $m = 2$ and consider the sequence (x_k) defined by $(x_k) = 1$ for *k* odd and $(x_k) = 2$ for *k* even. Consider the rearranged sequence (y_k) as

$$\left(y_k\right) = \left(x_1, x_3, x_2, x_4, x_5, x_7, x_6, x_8, \ldots\right)$$

Then $\left(y_k\right) \notin c_0\left(\Delta_2\right)$. Hence $c_o\left(\Delta_2\right)$ is not symmetric.

Next let $m = 1$ and consider the sequence (x_k) defined as $x_k = k$ for all $k \in N$. Consider its rearrangement defined as

$$(y_k) = \begin{pmatrix} x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, \\ x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots \end{pmatrix}$$

Then $(x_k) \in c(\Delta) \subset \ell_\infty(\Delta)$, but $(y_k) \notin \ell_\infty(\Delta)$. Hence the spaces $c(\Delta)$ and $\ell_\infty(\Delta)$ are not symmetric.

Theorem 8. *The spaces* $\ell_\infty(\Delta_m), c(\Delta_m)$ *and* $c_o(\Delta_m)$ *are not convergence free.*

Proof. The result follows from the following example.

Example 3. Let $m = 2$ and consider the sequence (x_k) defined as $x_k = 1$, for all $k \in N$. Then $(x_k) \in c_o(\Delta_2) \subset c(\Delta_2) \subset \ell_\infty(\Delta_2)$. Now consider the sequence (y_k) defined by $y_k = k^2$, for all $k \in N$, then $(y_k) \notin \ell_\infty(\Delta_2)$. Hence the spaces $c_o(\Delta_m)$, $c(\Delta_m)$ and $\ell_\infty(\Delta_m)$ are not convergence free .

Theorem 9. *The spaces* $\ell_\infty(\Delta_m), c(\Delta_m)$ *and* $c_o(\Delta_m)$ *are not monotone.*

Proof. The proof follows from the following example.

Example 4. Let $m = 1$ and consider the sequence $x = (x_k)$ defined as $x_k = 1$, for all $k \in N$. Then $(x_k) \in c_o(\Delta)$. Now consider the sequence (y_k) in its preimage space defined by $y_k = 1$, for k odd and by $y_k = 0$, for k even, then $(y_k) \notin c_o(\Delta)$. Hence the space $c_o(\Delta)$ is not monotone.

Next consider the sequence $x = (x_k)$ defined as $x_k = k$, for all $k \in N$. Then $(x_k) \in c(\Delta) \subset \ell_\infty(\Delta)$. Now consider the sequence (y_k) in its preimage space, defined as above, then $(y_k) \notin \ell_\infty(\Delta)$. Hence the spaces $c(\Delta_m)$ and $\ell_\infty(\Delta_m)$ are not monotone.

Theorem 10. $\ell_\infty(\Delta_m), c(\Delta_m)$ *and* $c_o(\Delta_m)$ *are* 1-*convex.*

Proof. If $0 \prec \delta \prec 1$, then

$$V = \left\{ x = (x_k) : \ \|x\|_{\Delta_m} \le \delta \right\}$$

is an absolutely 1-convex set, for let $x, y \in V$ and $|\lambda| + |\mu| \le 1$, then

$$\|\lambda x + \mu y\|_{\Delta_m} \le (|\lambda| + |\mu|)\delta \le \delta .$$

This completes the proof.

4. References

1. Cooke, R.G. (1950). *Infinite Matrices and Sequence Spaces;* MacMillan , London.
2. Kamthan, P.K. and Gupta, M.(1981):.Sequence Spaces and Series ; Marcel Dekker Inc. ; New York.
3. Kizmaz, H. (1981). On Certain Sequence Spaces, *Canad.Math.Bull.*; **24** 169-176.
4. Maddox, I.J.(1970). *Elements of Functional Analysis* Cambridge Univ.Press.
5. Maddox I.J.,and Roles, J.W.(1969). Absolute convexity in certain topological linear spaces, *Proc.Camb.Phil.Soc.*, **66**, 541-545.
6. Tripathy, B.C.(2003). A class of difference sequence sequences related to the p-normed space ℓ^p ; *Demonstratio Math.* ; **36**(4), 867-872.
7. Tripathy, B.C.(2003).On some class of difference paranormed sequence spaces associated with multiplier sequences ; *Internat. J. Math. Sci.*; *vol.***2**(*no*.1); 159-166.
8. Et,M.,.and Esi,A. (2000).On Köthe-Toeplitz duals of generalized difference sequence spaces; Bull. Malaysian Math.Sci.Soc. (Second Series), **23**, 1-8 .