A New Type Of Difference Sequence Spaces

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Abstract: In this paper we introduce the notion of the difference operator \( \Delta_m x_k \) for a fixed \( m \in \mathbb{N} \). We define the sequence spaces \( \ell^\infty (\Delta_m) \), \( c(\Delta_m) \) and \( c_0(\Delta_m) \) \((m \in \mathbb{N})\) and study some topological properties of these spaces. We obtain some inclusion relations involving these sequence spaces.

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Fark Dizi Uzaylarının Yeni Bir Şekli

Özet: Bu çalışmada sabit bir \( m \in \mathbb{N} \) sayısı için \( \Delta_m x_k \) fark operatörü yardımıyla \( \ell^\infty (\Delta_m) \), \( c(\Delta_m) \) ve \( c_0(\Delta_m) \) dizi uzayları tanımlanıp, bu uzaylar için bazı topolojik özellikler çalışılmış ve bu uzaylara ait bazı kapsam bağıntıları verilmiştir.

Anahtar Kelimeler: Fark dizi uzayı, Solid uzay, Simetrik uzay, Tamlık

1. Introduction

Throughout the paper \( w, \ell^\infty, c, \) and \( c_0 \) denote the spaces of all, bounded, convergent, and null sequences \( x = (x_k) \) with complex terms, respectively, normed by \(||x||_\infty = \sup_k |x_k|\).

The zero sequence is denoted by \( \theta = (0,0,0,\ldots) \).

Kizmaz [3] defined the difference sequence spaces \( \ell^\infty (\Delta), c(\Delta) \) and \( c_0(\Delta) \) as follows:

\[ Z(\Delta) = \{ x = (x_k) : (\Delta x_k) \Delta Z \}, \]

for \( Z = \ell^\infty, c \) and \( c_0 \), where \( \Delta x = (\Delta x_k) = (x_k - x_{k+1}) \), for all \( k \in \mathbb{N} \).

The above spaces are Banach spaces, normed by \(||x||_\Delta = |x_1| + \sup_k ||\Delta x_k||\).

The idea of Kizmaz [3] was applied to introduce different type of difference sequence spaces and study their different properties by Tripathy ([6], [7]), Et and Esi [8] and many others.

2. Definitions and Preliminaries

A sequence so ace \( E \) said to be solid (or normal) if \( (x_k) \in E \) implies \( (\alpha_k x_k) \in E \) for all sequences of scalars \( (\alpha_k) \) with \(|\alpha_k| \leq 1 \) for all \( n \in \mathbb{N} \).

A sequence space \( E \) is said to be monotone if it contains the canonical preimages of all its step spaces.

A sequence space \( E \) is said to be convergence free if \( (y_k) \in E \) whenever \( (x_k) \in E \) and \( y_k = 0 \) whenever \( x_k = 0 \).

A sequence space \( E \) is said to be a sequence algebra if \( (x_k y_k) \in E \) whenever \( (x_k) \in E \) and \( (y_k) \in E \).

A sequence space \( E \) is said to be symmetric if \( (x_{\pi(k)}) \in E \) whenever \( (x_k) \in E \), where \( \pi(k) \) is a permutation on \( \mathbb{N} \).

For \( r > 0 \), a nonempty subset \( V \) of a linear space is said to be absolutely \( r\)-convex if \( x, y \in V \)
and $|\lambda|^r + |\mu|^r \leq 1$ together imply that

$\lambda x + \mu y \in V$. A linear topological space $X$ is said
to be $r$-convex if every neighborhood of $\theta \in X$
contains as absolutely $r$-convex neighborhood of $\theta \in X$ (see for instance Maddox and Roles [5]).

Let $m \in N$ be fixed, then we introduce the
following new type of difference sequence spaces

$$Z(\Delta_m) = \{x = (x_k) \in w: \Delta_m x \in Z\},$$

for $Z = \ell_{\infty}, c$, and $c_0$, where

$$\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$$

for all $k \in N$.

For $m = 1$, $\ell_{\infty}(\Delta_m) = \ell_{\infty}(\Delta)$, $c(\Delta_m) = c(\Delta)$ and $c_0(\Delta_m) = c_0(\Delta)$. Hence the introduced notion generalizes
the notion of difference sequences studied by Kizmaz [3].

3. Main Results

In this section we establish the results of this
article. The proof of the following result is a routine
verification.

Proposition 1. The classes of sequences

$$\ell_{\infty}(\Delta_m), c(\Delta_m) \text{ and } c_0(\Delta_m)$$

are normed linear spaces, normed by

$$\|x\|_{\lambda_m} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k|$$

(1)

Proof. Let $\alpha, \beta$ be scalars and $(x_k), (y_k) \in \ell_{\infty}(\Delta_m)$.

Then

$$\sup_k |\Delta_m x_k| < \infty \text{ and } \sup_k |\Delta_m y_k| < \infty$$

(2)

Hence

$$\sup_k |\Delta_m (\alpha x_k + \beta y_k)| \leq |\alpha| \sup_k |\Delta_m x_k| + |\beta| \sup_k |\Delta_m y_k| < \infty, \text{ by (2).}$$

Hence $\ell_{\infty}(\Delta_m)$ is a linear space. Similarly it can
be shown that $c(\Delta_m)$ and $c_0(\Delta_m)$ are linear spaces.

Next for $x = 0$, we have $\|\theta\|_{\lambda_m} = 0$. Conversely,
let $\|x\|_{\lambda_m} = 0$. Then

$$\|x\|_{\lambda_m} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k| = 0.$$

$\Rightarrow x_r = 0$ for $r = 1, 2, \ldots, m$ and $|\Delta_m x_k| = 0$ for
all $k \in N$.

Consider $k = 1$ i.e. $|\Delta_m x_1| = 0 \Rightarrow |x_1 - x_{1+m}| =
\Rightarrow x_{m+1} = 0$, since $x_m = 0$.

Proceeding in this way we have $x_k = 0$, for all $k \in N$.

$$\|x + y\|_{\lambda_m} = \sum_{r=1}^{m} |x_r + y_r| + \sup_k |\Delta_m (x_k + y_k)|$$

$$\leq \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k| + \sum_{r=1}^{m} |y_r| + \sup_k |\Delta_m y_k|$$

$$= \|x\|_{\lambda_m} + \|y\|_{\lambda_m}$$

Finally

$$||\lambda x||_{\lambda_m} = \sum_{r=1}^{m} |\lambda x_r| + \sup_k |\Delta_m (\lambda x_k)|$$

$$= |\lambda| \|x\|_{\lambda_m}.$$"
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\[ \left\| x^n - x^i \right\|_{\infty} = \sum_{r=1}^{m} |x^n_r - x^i_r| + \sup_k \left| \Delta_m x^n_k - \Delta_m x^i_k \right| < \varepsilon \]

, for all \( n, i \geq n_0 \). - - - (3)

Hence \( |x^n_k - x^i_k| < \varepsilon \) for all \( n, i \geq n_0 \) and \( k = 1, \ldots, m \).

\[ \Rightarrow \left( x^n_k \right) \text{ is a Cauchy sequence in } C \text{ for } k = 1, 2, \ldots, m. \]

\[ \Rightarrow \left( x^i_k \right) \text{ is a convergent in } C \text{ for } k = 1, 2, \ldots, m. \]

Let \( \lim_{i \to \infty} x^i_k = x_k \), say for \( k = 1, 2, \ldots, m \).

From (3) we have \( |\Delta_m x^n_k - \Delta_m x^i_k| < \varepsilon \), for all \( n, i \geq n_0 \) and all \( k \in N \). Hence \( \left( \Delta_m x^n_k \right) \) is a Cauchy sequence in \( C \) for all \( k \in N \). Thus \( \left( \Delta_m x^n_k \right) \) is convergent in \( C \), let \( \lim_{i \to \infty} \Delta_m x^i_k = y_k \), say for each \( k \in N \). Since \( \lim_{i \to \infty} x^i_k = x_k \), exists for \( k = 1, 2, \ldots, m \), so we have \( \lim_{i \to \infty} x^i_k = x_k \), exists for each \( k \in N \).

We have

\[ \lim_{i \to \infty} \sum_{i=1}^{m} |x^n_i - x_i| = \sum_{i=1}^{m} |x^n_i - x_i| < \varepsilon, \]

for all \( i \geq n_0 \), and

\[ \lim_{i \to \infty} |x^n_k - x^i_k| \leq |x^n_k - (x^i_{k+m} - x^i_{k+m})|, \]

for all \( k \in N \) and \( n \geq n_0 \).

Hence for all \( i \geq n_0 \), we have

\[ \sup_k |\Delta x^n_k - \Delta x^i_k| < \varepsilon. \]

Thus

\[ \sum_{r=1}^{m} |x^n_r - x_r| + \sup_k |\Delta x^n_k - \Delta x^i_k| < 2\varepsilon, \]

\[ \Rightarrow (x^n - x) \in \ell_\infty(\Delta_m), \text{ for all } i \geq n_0. \]

Thus \( x = x^n - (x^n - x) \in \ell_\infty(\Delta_m), \text{ for all } i \geq n_0. \)

since \( \ell_\infty(\Delta_m) \) is a linear space.

Hence \( \ell_\infty(\Delta_m) \) is complete.

Similarly it can be shown that the spaces \( c(\Delta_m) \) and \( c_o(\Delta_m) \) are also complete.

The following result is a consequence of the above result and the definition of BK-space.

**Proposition 4.** The spaces \( \ell_\infty(\Delta_m), c(\Delta_m) \) and \( c_o(\Delta_m) \) are BK-spaces.

Since the inclusions \( c(\Delta_m) \subset \ell_\infty(\Delta_m) \) and \( c_o(\Delta_m) \subset \ell_\infty(\Delta_m) \) are proper, the following result follows from Theorem 3.

**Proposition 5.** The spaces \( c(\Delta_m) \) and \( c_o(\Delta_m) \) are nowhere dense subsets of \( \ell_\infty(\Delta_m) \).

**Theorem 6.** The spaces \( \ell_\infty(\Delta_m), c(\Delta_m) \) and \( c_o(\Delta_m) \) are not solid spaces.

**Proof.** The proof follows from the following examples.

**Example 1.** Let \( x_k = k \) for all \( k \in N \). Consider the sequence of scalars \( (\alpha_k) \) defined by \( \alpha_k = im + 1 \), for \( i = 1, 2, \ldots, \) and \( \alpha_k = 0 \), otherwise. Then \( (x_k) \in c(\Delta_m) \subset \ell_\infty(\Delta_m) \), but \( (\alpha_k x_k) \notin \ell_\infty(\Delta_m) \).

Hence the spaces \( c(\Delta_m) \) and \( \ell_\infty(\Delta_m) \) are not solid.

For the case \( c_o(\Delta_m) \), consider the sequence \( x_k \) = 1 for all \( k \in N \) and the sequence \( (\alpha_k) \) defined as above. Then \( (x_k) \in c_o(\Delta_m) \), but \( (\alpha_k x_k) \notin c_o(\Delta_m) \).

Hence \( c_o(\Delta_m) \) is not solid.

**Theorem 7.** (i) The space \( c_o(\Delta) \) is symmetric.

(ii) The spaces \( \ell_\infty(\Delta_m), c(\Delta_m) \) and \( c_o(\Delta_m) \) (for \( m > 1 \)) are not symmetric spaces.

**Proof.** (i) The first part is known. For the second part, consider the following example.

**Example 2.** Let \( m = 2 \) and consider the sequence \( (x_k) \) defined by \( x_k = 1 \) for \( k \) odd and \( x_k = 2 \) for \( k \) even. Consider the rearranged sequence \( (y_k) \) as \( y_k = \begin{cases} x_k & \text{if } k \text{ odd} \\ x_{k+1} & \text{if } k \text{ even} \end{cases} \). Then \( (y_k) \notin c_o(\Delta_2) \). Hence \( c_o(\Delta_2) \) is not symmetric.

Next let \( m = 1 \) and consider the sequence \( (x_k) \) defined as \( x_k = k \) for all \( k \in N \). Consider its rearrangement defined...
as
\[
(y_k) = \left\{ x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, \right. \\
\left. x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots \right\}
\]

Then \( (x_k) \in c(\Delta) \subset \ell_{\infty}(\Delta) \), but \( (y_k) \not\in \ell_{\infty}(\Delta) \).
Hence the spaces \( c(\Delta) \) and \( \ell_{\infty}(\Delta) \) are not symmetric.

**Theorem 8.** The spaces \( \ell_{\infty}(\Delta_m) \), \( c(\Delta_m) \) and \( c_o(\Delta_m) \) are not convergence free.

**Proof.** The result follows from the following example.

**Example 3.** Let \( m = 2 \) and consider the sequence \((x_k)\) defined as \( x_k = 1 \), for all \( k \in \mathbb{N} \). Then \( (x_k) \in c_o(\Delta_2) \subset c(\Delta_2) \subset \ell_{\infty}(\Delta_2) \). Hence the spaces \( c_o(\Delta_m) \), \( c(\Delta_m) \) and \( \ell_{\infty}(\Delta_m) \) are not convergence free.

**Theorem 9.** The spaces \( \ell_{\infty}(\Delta_m) \), \( c(\Delta_m) \) and \( c_o(\Delta_m) \) are not monotone.

**Proof.** The proof follows from the following example.

**Example 4.** Let \( m = 1 \) and consider the sequence \( x = (x_k) \) defined as \( x_k = 1 \), for all \( k \in \mathbb{N} \). Then \( (x_k) \in c_o(\Delta) \). Now consider the sequence \((y_k)\) in its preimage space defined by \( y_k = 1 \), for \( k \) odd and by \( y_k = 0 \), for \( k \) even, then \( (y_k) \not\in c_o(\Delta) \). Hence the space \( c_o(\Delta) \) is not monotone.

Next consider the sequence \( x = (x_k) \) defined as \( x_k = k \), for all \( k \in \mathbb{N} \). Then \( (x_k) \in c(\Delta) \subset \ell_{\infty}(\Delta) \). Now consider the sequence \((y_k)\) in its preimage space, defined as above, then \( (y_k) \not\in \ell_{\infty}(\Delta) \). Hence the spaces \( c(\Delta_m) \) and \( \ell_{\infty}(\Delta_m) \) are not monotone.

**Theorem 10.** \( \ell_{\infty}(\Delta_m) \), \( c(\Delta_m) \) and \( c_o(\Delta_m) \) are 1-convex.

**Proof.** If \( 0 < \delta < 1 \), then \( V = \left\{ x = (x_k) : \left\| x \right\|_{\Delta_m} \leq \delta \right\} \) is an absolutely 1-convex set, for let \( x, y \in V \) and \( |\lambda| + |\mu| \leq 1 \), then
\[
\left\| \lambda x + \mu y \right\|_{\Delta_m} \leq (|\lambda| + |\mu|) \delta \leq \delta.
\]
This completes the proof.

4. References