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On New Difference Sequence Spaces Generated by Infinite Matrices

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Abstract: The main purpose of the present paper is to introduce the spaces $\left(\hat{A}, p, \Delta^2\right)_0$, $\left(\hat{A}, p, \Delta^2\right)$ and $\left(\hat{A}, p, \Delta^2\right)_\infty$ consisting of all sequences whose differences are in the spaces $\left(\hat{A}, p, \Delta\right)_0$, $\left(\hat{A}, p, \Delta\right)$ and $\left(\hat{A}, p, \Delta\right)_\infty$, respectively, and to fill up the gap in the existing literature. Also we investigate some properties of these spaces.

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Keywords: Sequence space, Linear topological space, Paranorm

Sonsuz Matrisler Tarafından Üretilen Yeni Fark Dizi Uzayları Üzerine

Özet: Bu çalışmanın esas amacı, farkları $\left(\hat{A}, p, \Delta\right)_0$, $\left(\hat{A}, p, \Delta\right)$ ve $\left(\hat{A}, p, \Delta\right)_\infty$ dizi uzaylarında olan, $\left(\hat{A}, p, \Delta^2\right)_0$, $\left(\hat{A}, p, \Delta^2\right)$ ve $\left(\hat{A}, p, \Delta^2\right)_\infty$ fark dizi uzaylarını tanımlayıp bazı özelliklerini inceleyerek literatürdeki bir boşluğu doldurmaktır.

2000 Matematik Konu Sınıflandırması: 46A45.

Anahtar Kelimeler: Dizi uzayı, Lineer topolojik uzay, Paranorm

1. Introduction

Let l_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\|_\infty = \sup_{k\geq 0}|x_k|$. Kızmaz [1] defined the sequence spaces

$$c_0(\Delta) = \{x : \Delta x \in c_0\}$$
$$c(\Delta) = \{x : \Delta x \in c\}$$
$$l_\infty(\Delta) = \{x : \Delta x \in l_\infty\}$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. These spaces are Banach with norm $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$.

Furthermore, the sequence spaces $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta l_\infty(p)$ have been defined by Ahmad and Mursaleen [2] as follows

$$\Delta c_0(p) = \{x : \Delta x \in c_0(p)\}$$
$$\Delta c(p) = \{x : \Delta x \in c(p)\}$$
$$\Delta l_\infty(p) = \{x : \Delta x \in l_\infty(p)\}.$$

After Et [3] defined the sequence spaces

$$c_0(\Delta^2) = \{x : \Delta^2 x \in c_0\}$$
$$c(\Delta^2) = \{x : \Delta^2 x \in c\}$$
$$l_\infty(\Delta^2) = \{x : \Delta^2 x \in l_\infty\}$$

where $\Delta^2 x = (\Delta^2 x_k) = (\Delta x_k - \Delta x_{k+1})$. The sequence spaces $l_\infty(\Delta^2)$, $c(\Delta^2)$ and $c_0(\Delta^2)$ are Banach spaces with norm $\|x\|_\Delta = |x_1| + |x_2| + \|\Delta^2 x\|_\infty$.

Let $A = (a_{nk})$ be an infinite matrix of nonnegative real numbers and (p_k) be a bounded sequence of positive real numbers. (These assumptions are made throughout). We write $B_{mn}(x) = \sum_k a_{mk} x_{k+n}$ if the series converges for each m and n. Here and afterwards summation without limits runs from 1 to ∞.

Nanda [4] defined the sequence spaces

$$(\hat{A},p)_0 = \{x : |B_{mn}(x)|^{p_m} \to 0 \text{ as } m \to \infty \text{ uniformly in } n\}$$

$$(\hat{A},p) = \{x : |B_{mn}(x-le)|^{p_m} \to 0 \text{ as } m \to \infty \text{ for some } l \text{ uniformly in } n\}$$

$$(\hat{A},p)_\infty = \left\{x : \sup_{m,n} |B_{mn}|^{p_m} < \infty\right\}$$

Finally, the sequence spaces $(\hat{A},p,\Delta)_0$, $(\hat{A},p,\Delta)$, $(\hat{A},p,\Delta)_\infty$ have been defined by Solak [5] as follows

$$(\hat{A},p,\Delta)_0 = \{x : |\Delta B_{mn}(x)|^{p_m} \to 0 \text{ as } m \to \infty \text{ uniformly in } n\}$$

$$(\hat{A},p,\Delta) = \{x : |\Delta B_{mn}(x-le)|^{p_m} \to 0 \text{ as } m \to \infty \text{ for some } l \text{ uniformly in } n\}$$

$$(\hat{A},p,\Delta)_\infty = \left\{x : \sup_{m,n} |\Delta B_{mn}(x)|^{p_m} < \infty\right\}$$

where $\Delta B_{mn}(x) = \sum_k \Delta a_{mk} x_{k+n}$, $(\Delta a_{mk} = a_{mk} - a_{m+1,k})$.

Now, we introduce new sequence spaces $(\hat{A},p,\Delta^2)_0$, $(\hat{A},p,\Delta^2)$, $(\hat{A},p,\Delta^2)_\infty$ and investigate some properties these spaces.

Now we define

$$(\hat{A},p,\Delta^2)_0 = \{x : |\Delta^2 B_{mn}(x)|^{p_m} \to 0 \text{ as } m \to \infty \text{ uniformly in } n\}$$

$$(\hat{A},p,\Delta^2) = \{x : |\Delta^2 B_{mn}(x-le)|^{p_m} \to 0 \text{ as } m \to \infty \text{ for some } l \text{ uniformly in } n\}$$

$$(\hat{A},p,\Delta^2)_\infty = \left\{x : \sup_{m,n} |\Delta^2 B_{mn}(x)|^{p_m} < \infty\right\}$$

where $\Delta^2 B_{mn}(x) = \sum_k \Delta^2 a_{mk} x_{k+n}$,

$(\Delta^2 a_{mk} = \Delta(\Delta a_{mk}) = a_{mk} - 2a_{m+1,k} + a_{m+2,k})$.

2. Linear Topological Structure of These Spaces

We assume through that $0 < p_m \le 1$, for if $0 < p_m < \infty$ and $\sup p_m < \infty$ then $0 < \frac{p_m}{\sup p_m} \le 1$ and we can, without loss of generality, replace p_m by $\frac{p_m}{\sup p_m}$. It is clear that $(\hat{A},p,\Delta^2)_0 \subset (\hat{A},p,\Delta^2)$ and $(\hat{A},p,\Delta^2)_0 \subset (\hat{A},p,\Delta^2)_\infty$. But we have been able to prove $(\hat{A},p,\Delta^2) \subset (\hat{A},p,\Delta^2)_\infty$ only for a special case. We have

Theorem 1. $(\hat{A},p,\Delta^2) \subset (\hat{A},p,\Delta^2)_\infty$ *if*

$$\sup_m \left|\sum_k \Delta^2 a_{mk}\right|^{p_m} < \infty \tag{2.1}$$

Proof. Suppose that $x \in (\hat{A},p,\Delta^2)$ and (2.1) holds. We have

$$\begin{aligned} |\Delta^2 B_{mn}(x)|^{p_m} &= |\Delta^2 B_{mn}(x-le+le)|^{p_m} \\ &\le |\Delta^2 B_{mn}(x-le)|^{p_m} \\ &\quad + \left|l \sum_k \Delta^2 a_{mk}\right|^{p_m} \qquad (2.2) \\ &\le |\Delta^2 B_{mn}(x-le)|^{p_m} \\ &\quad + \sup |l|^{p_m} \left|\sum_k \Delta^2 a_{mk}\right|^{p_m} \end{aligned}$$

Therefore $x \in (\hat{A},p,\Delta^2)_\infty$ and this completes proof.

A paranormed space $X = (X,g)$ is a topological linear space in which the topology is given by paranorm g a real subadditive function on X such that $g(\theta) = 0$, $g(x) = g(-x)$ and such that multiplication is continuous. In the above, θ is zero in the complex linear space X and continuity of multiplication means that $\lambda_n \to \lambda$, $x_n \to x$ imply $\lambda_n x_n \to \lambda x$ for scalar λ and $x \in X$. We have

Theorem 2. $(\hat{A},p,\Delta^2)_0$ *is a linear topological space paranormed by* g *defined by*

$$g(x) = \sup_{m,n} |\Delta^2 B_{mn}(x)|^{p_m}.$$

$(\hat{A},p,\Delta^2)_\infty$ *is paranormed by* g *if* $\inf p_m > 0$. *If* (2.1) *holds,* $(\hat{A},p,\Delta^2)$ *is paranormed with the same paranorm* g. $(\hat{A},p,\Delta^2)_0$ *and* $(\hat{A},p,\Delta^2)_\infty$ *are complete in their paranorm topologies.* $(\hat{A},p,\Delta^2)$ *is complete if*

$$\left|\sum_k \Delta^2 a_{mk}\right| \to 0 \text{ as } m \to \infty \tag{2.3}$$

Proof. This is routine verication and can be obtained by using standart techniques. If (2.1) holds, then by Theorem1 $(\hat{A},p,\Delta^2) \subset (\hat{A},p,\Delta^2)_\infty$

and hence $\left(\hat{A}, p, \Delta^2\right)$ has the same paranorm g If (2.3) holds, then (2.1) also holds and it follows from the inequality (2.2), that $\left(\hat{A}, p, \Delta^2\right) = \left(\hat{A}, p, \Delta^2\right)_0$ and therefore the completeness of $\left(\hat{A}, p, \Delta^2\right)$ follows from the completeness of $\left(\hat{A}, p, \Delta^2\right)_0$.

Theorem 3. *If* $0 < p_m < q_m \leq 1$*, then* $\left(\hat{A}, q, \Delta^2\right)_\infty$ *is a closed subspace of* $\left(\hat{A}, p, \Delta^2\right)_\infty$.

Proof. Let $x \in \left(\hat{A}, q, \Delta^2\right)_\infty$. Then there exists a constant H such that $\left|\Delta^2 B_{mn}(x)\right|^{p_m} \leq H$ for all m, n. This implies that $\left|\Delta^2 B_{mn}(x)\right|^{p_m} \leq H$ for all m, n .Thus $x \in \left(\hat{A}, p, \Delta^2\right)_\infty$. To show that $\left(\hat{A}, q, \Delta^2\right)_\infty$ is closed, suppose that $x^i \in \left(\hat{A}, q, \Delta^2\right)_\infty$, $x^i \to x$ and $x \in \left(\hat{A}, p, \Delta^2\right)_\infty$. Then for every ε, $0 < \varepsilon < 1$, there exist for all m, n

$$\left|\Delta^2 B_{mn}\left(x^i - x\right)\right|^{p_m} < \varepsilon \text{ for } i > N.$$

Now

$$\left|\Delta^2 B_{mn}\left(x^i - x\right)\right|^{q_m} < \left|\Delta^2 B_{mn}\left(x^i - x\right)\right|^{p_m} < \varepsilon \text{ for } i > N.$$

Therefore $x \in \left(\hat{A}, q, \Delta^2\right)_\infty$ and this completes the proof.

3. References

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