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On A Class Of Statistically Null Vector Valued Sequences Associated With Multilpier Sequences

Mausumi SEN and Binod Chandra TRIPATHY
Mathematical Sciences Division Institute of Advanced Study in Science and Technology
Paschim Boragaon; GARCHUK; GUWAHATI – 781 035; INDIA.
tripathybc@yahoo.com

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Abstract : In this article we introduce the multiplier vector valued sequence space $\bar{c}_0\{E_k, \Lambda, p\}$, where $\Lambda = (\gamma_k)$ is an associated multiplier sequence of non-zero complex numbers and the terms of the sequence are chosen from the semi normed spaces E_k, $k \in N$. We study some properties of these spaces like completeness, solidity, inclusion of two different classes. We prove the decomposition theorem.

Keywords: Paranorm, solid space, multiplier sequence, density.

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1. Introduction

The notion of statistical convergence of sequences was introduced by Fast[2] and Schoenberg [13]. It is also found in zygmund[17]. Later on it was studied from sequence space point of view and linked with summability by Fridy [3], Connor[1], Šalàt [12], Tripathy[15] and many others. The idea depends on the density of subsets of the set N of natural numbers. A subset E of N is said to have density $\delta(E)$ if

$\delta(E) = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \chi_A(k)$ exists, where χ_E is the characteristic function of E.

For (x_k) and (y_k) two sequences, we say that $x_k = y_k$ for almost all k (in short *a.a.k*) if $\delta(\{k \in N : x_k \neq y_k\}) = 0$.

A sequence (x_k) is said to be *statistically convergent* to L if for any $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$ and we write stat-lim $x_k = L$.

A sequence (x_k) is said to be *statistically. Cauchy* provided that for every $\varepsilon > 0$, there exists a number n_0 such that $|x_k - x_{n_0}| < \varepsilon$ for *a.a.k*.

The scope for the studies on sequences spaces was further extended by using the notion of associated multiplier sequences. Goes and Goes [4] defined the differentiated sequence space dE and the integrated sequence space $\int E$ for a given sequence space E, using multiplier sequences (k^{-1}) and (k) respectively. Kamthan [5] used the multiplier sequence $(k!)$. In the present article we shall consider a general multiplier sequence $\Lambda = (\gamma_k)$ of non-zero scalars.

Throughout ℓ_∞, c, c_0 and $\bar{c}_0$ denote the spaces of bounded, convergent, null and statistically null sequences respectively.

2. Definitions And Preliminaries

The concept of paranormed sequence space was studied by Nakano [10] and Simons [14] at the initial stage. Later on it was studied by Maddox [9], Lascarides [7] , and many others.

The studies on vector valued sequence spaces was explored by Ratha and Srivastava [11], Leonard [8] and many others.

Throughout the article E_k will denote a seminormed sequence space, seminormed by f_k for all $k \in N$, defined over C, the field of complex numbers. Throughout $p = (p_k)$ is a sequence of strictly positive real numbers and $t_k = p_k^{-1}$, for all $k \in N$.

A vector valued sequence space E is said to be *solid* (or *normal*) if $\alpha x = (\alpha_k x_k) \in E$, whenever $x = (x_k) \in E$ and $\alpha = (\alpha_k)$ is a sequence of scalars such that $|\alpha_k| \le 1$, for all $k \in N$.

A sequence space E is said to be *monotone* if E contains the canonical preimages of all its subspaces (one may refer to Kamthan and Gupta [6], p. 48).

A sequence space E is said to be *symmetric* if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of N.

Tripathy and Sen [16] have studied the following vector valued sequence spaces :

$\ell_\infty(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k$ for all $k \in N$ and $\sup_k (f_k(\gamma_k x_k))^{p_k} < \infty \}$

$c_0(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k$ for all $k \in N$ and $(f_k(\gamma_k x_k))^{p_k} \to 0$, as $k \to \infty \}$

$\ell_\infty\{E_k, \Lambda, p\} = \{ (x_k) : x_k \in E_k$ for all $k \in N$ and there exists $r > 0$ such that

$\sup_k (f_k(r\gamma_k x_k))^{p_k} t_k < \infty \}$ $c_0\{E_k, \Lambda, p\} = \{ (x_k) : x_k \in E_k$ for all $k \in N$ and there exists

$r > 0$ such that
$(f_k(r\gamma_k x_k))^{p_k} t_k \to 0$, as $k \to \infty \}$

If $p \in \ell_\infty$ and each E_k is complete, then $c_0\{E_k, \Lambda, p\}$ is a complete paranormed space, paranormed by $g(x) = \sup_k (f_k(\gamma_k x_k p^{-t_k}))^{\frac{p_k}{M}}$, where $M = max(1, H)$, $H = \sup_k p_k$.

The space $\ell_\infty\{E_k, \Lambda, p\}$ is paranormed by g if $\inf p_k > 0$.

In this article we introduce the following class of vector valued sequences associated with multiplier sequences :

$\bar{c}_0(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k$ for all $k \in N$ and $(f_k(\gamma_k x_k))^{p_k} t_k \xrightarrow{stat} 0$, as $k \to \infty \}$,

$\bar{c}_0\{E_k, \Lambda, p\} = \{ (x_k) : x_k \in E_k$ for all $k \in N$ and there exists $r > 0$ such that

$(f_k(\gamma_k x_k r))^{p_k} t_k \xrightarrow{stat} 0$, as $k \to \infty$ }.

We define

$$m_0(E_k, \Lambda, p) = \bar{c}_0(E_k, \Lambda, p) \cap \ell_\infty(E_k, \Lambda, p)$$

and

$$m_0\{E_k, \Lambda, p\} = \bar{c}_0\{E_k, \Lambda, p\} \cap \ell_\infty\{E_k, \Lambda, p\}.$$

Let $\delta_0\{E_k\} = \{ (x_k) : x_k \in E_k$ for all $k \in N$ and $\delta(\{ k \in N : x_k = \theta_k \}) = 1\}$,
where θ_k is the zero element of E_k.

Two sequence spaces E and F are said to be *equivalent* if there exists a sequence $u = (u_k)$ of strictly positive numbers such that the mapping $u : E \to F$ defined by $y = ux = (u_k x_k) \in F$, whenever $(x_k) \in E$, is a one-to-one correspondence between E and F. It is denoted by $E \cong F(u)$ or simply $E \cong F$ (see for instance Nakano [10]).

It is remarked by Lascarides ([7] Remark 3) that " *If E is a sequence space paranormed* (or *normed*) *by g and $E \cong F(u)$, then F is a sequence space paranormed* (or *normed*) *by g_u defined by* $g_u(y) = g(u^{-1}y)$, $y \in F$."

Further it is noted by Lascarides [7] that " if $(p_k) \in \ell_\infty$, then $c_0(p) \cong c_0\{p\}(u)$, (as well as $\ell_\infty(p) \cong \ell_\infty\{p\}(u)$), where $u = (p_k^{t_k})$. "

For E, F two sequence spaces we define $M(F, E)$ as follows:

$M(F, E) = \{ (\gamma_k) : (\gamma_k x_k) \in E$ for all $(x_k) \in F \}$,

where $\Lambda = (\gamma_k)$ is a multiplier sequence.

Lemma 1. *A sequence space E is solid implies that E is monotone.*

Lemma 2. (Fridy [3], Theorem 1) . *The following statements are equivalent* :

(i) (x_k) *is a statistically convergent sequence*

(ii) (x_k) *is statistically Cauchy sequence*

(iii) (x_k) *is a sequence for which there is a convergent sequence* (y_k) *such that*

$$x_k = y_k \text{ for a.a.}k .$$

Lemma 3. (Šalàt [12], Lemma 1.1) . *A sequence* (x_k) *statistically converges to L if and only if there exists such a set*
$K = \{ k_1 < k_2 < k_3 < \ldots \} \subset N$ that $\delta(K) = 1$ and $\lim_{n\to\infty} x_{k_n} = L$.

Lemma 4. (Connor [1], Theorem 2.3) . *If* $x = (x_k)$ *is statistically convergent to L, then there is a convergent sequence* $y = (y_k)$ *and a statistically null sequence* $z = (z_k)$ *such that y is convergent to L,* $x = y + z$ *and* $\delta(\{ k \in N : z_k \neq 0 \}) = 0$. *Moreover, if x is bounded then z is bounded and* $\| z \|_\infty \leq \| x \|_\infty + | L |$.

Lemma 5. (Lascarides [7], Proposition 1) . *Let* $h = \inf p_k$ *and* $H = \sup p_k$. *Then the following conditions are equivalent:*

(i) $H < \infty$ *and* $h > 0$

(ii) $c_0(p) = c_0$ or $\ell_\infty(p) = \ell_\infty$

(iii) $\ell_\infty\{p\} = \ell_\infty(p)$

(iv) $c_0\{p\} = c_0(p)$

(v) $\ell\{p\} = \ell(p)$.

3. Main Results

In this section we prove the results of this article.

Theorem 1. $\bar{c}_0\{E_k, \Lambda, p\}$ *is a linear space for any sequence* $p = (p_k)$.

Proof. Let $x \in \bar{c}_0\{E_k, \Lambda, p\}$. Then there exists $r > 0$ such that
$(f_k(\gamma_k x_k r))^{p_k} t_k \xrightarrow{stat} 0$, as $k \to \infty$.

Let $\xi \in C$ and without loss of generality let $\xi \neq 0$. Let $\rho = r\, |\, \xi\, |^{-1} > 0$, then we have
$(f_k(\gamma_k(\xi x_k))\rho))^{p_k} t_k =$
$(f_k(\gamma_k x_k r))^{p_k} t_k \xrightarrow{stat} 0$, as $k \to \infty$.

Therefore $\xi x \in \bar{c}_0\{E_k, \Lambda, p\}$, for all $\xi \in C$ and $x \in \bar{c}_0\{E_k, \Lambda, p\}$.

Next we suppose that $x, y \in \bar{c}_0\{E_k, \Lambda, p\}$. Then there exists $r_1 > 0$ and $r_2 > 0$ such that

$$(f_k(\gamma_k x_k r_1))^{p_k} t_k \xrightarrow{stat} 0 \text{ as } k \to \infty$$

and

$$(f_k(\gamma_k x_k r_2))^{p_k} t_k \xrightarrow{stat} 0 \text{ as } k \to \infty.$$

Then for a given $\varepsilon > 0$, we have
$\delta(K_1) = \delta(\{ k \in N : (f_k(r_1 \gamma_k x_k))^{p_k} t_k < \varepsilon \}) = 1$
and
$\delta(K_2) = \delta(\{ k \in N : (f_k(r_2 \gamma_k x_k))^{p_k} t_k < \varepsilon \}) = 1$

Let $r = (r_1 r_2 (r_1 + r_2)^{-1}$ and $K = K_1 \cap K_2$. Then clearly $\delta(K) = 0$ and for all $k \in K$, we have
$(f_k(r\gamma_k(x_k + y_k)))^{p_k} t_k \leq$
$[(f_k(r_1 \gamma_k x_k) r_2 (r_1 + r_2)^{-1})^{p_k} t_k +$
$(f_k(r_2 \gamma_k x_k) r_1 (r_1 + r_2)^{-1})^{p_k} t_k < \varepsilon\, p_k$.

Hence $x + y \in \bar{c}_0\{E_k, \Lambda, p\}$.

Since $\ell_\infty\{E_k, \Lambda, p\}$ is a linear space for any p , we have the following result.

Corollary 1. $m_0\{E_k, \Lambda, p\}$ *is a linear space for any sequence* $p = (p_k)$.

The proofs of the following two results are easy, so omitted.

Theorem 2. *Let* $0 < inf\ p_k \le p_k \le sup\ p_k$ *and each* E_k *is complete, then* $m_0\{E_k, \Lambda, p\}$ *is a complete paranormed space, paranormed by* g

Proposition 3. *The space* $m_0\{E_k, \Lambda, p\}$ *is solid as well as monotone.*

Proposition 4. *The space* $m_0\{E_k, \Lambda, p\}$ *is not symmetric.*

Proof. The result follows from the following example.

Let $E_k = c$, $p_k = 1$ and $\gamma_k = 1$, for all $k \in N$. Also let $f_k(x_k) = \sup_i |x_k^i|$,

where $x_k = (x_k^i)_i$, for all $k \in N$.

We consider the sequence (x_k) defined by

$x_k = e$, if $k = i^2, i \in N$

$= \bar{\theta}$, otherwise.

where $e = (1, 1, \text{-----})$ and $\bar{\theta} = (0, 0, \text{-----})$.

Then $(x_k) \in m_0\{E_k, \Lambda, p\}$. We consider the rearrangement (y_k) of (x_k) as

$y_k = e$, if k is odd

$= \bar{\theta}$, otherwise.

Then $(y_k) \notin m_0\{E_k, \Lambda, p\}$. Hence $m_0\{E_k, \Lambda, p\}$ is not symmetric.

Theorem 5. *If* $p \in \ell_\infty$, *then the following are equivalent*

(i) $(x_k) \in \bar{c}_0\{E_k, \Lambda, p\}$

(ii) *there exists a subset* $K = \{\{k_1, k_2, k_3, \ldots\}$ *of* N *such that* $\delta(K) = 1$ *and*

$$\lim_{i\to\infty} ((f_k(\gamma_{k_i} x_{k_i} r))^{p_{k_i}} t_{k_i} = 0.$$

(iii) *there exists* $(y_k) \in c_0\{E_k, \Lambda, p\}$ *such that* $x_k = y_k$ *for a.a.k.*

(iv) *there exists sequences* (y_k) *and* (z_k) *such that* $x_k = y_k + z_k$ *for all* $k \in N$ *and*

$$(y_k) \in c_0\{E_k, \Lambda, p\},\ (z_k) \in \delta_0(E_k).$$

Proof. Let $(x_k) \in \bar{c}_0\{E_k, \Lambda, p\}$. Then there exists $r>0$ such that $(f_k(\gamma_k x_k r))^{p_k} t_k \xrightarrow{stat} 0$ as $k \to \infty$. Let us consider $a_k = (f_k(\gamma_k x_k r))^{p_k} t_k$. Then $(a_k) \in \bar{c}_0$. The equivalence of (*i*) and (*ii)* follows from lemma 3, that of (*i*) and (*iii*) from lemma 2 and that of (*i*) and (*iv*) from lemma 4.

Theorem 6. *Let* (p_k) *be a given sequence of strictly positive numbers. Then* $(\gamma_k) \in (E, E)$ *if and only if* $((\gamma_k)^{p_k}) \in \ell_\infty$, *where* $E = m_0\{E_k, p\}$.

Proof. The sufficiency is obvious.

For the necessity, suppose that $((\gamma_k)^{p_k}) \notin \ell_\infty$.

Then there exists a subsequence $((\gamma_{k_i})^{p_{k_i}})$ of $((\gamma_k)^{p_k})$ such that

$$(\gamma_{k_i})^{p_{k_i}} \to \infty, \quad \text{as } i \to \infty.$$

Then we can find a sequence $(x_k) \in m_0\{E_k, p\}$ such that

$$|(\gamma_{k_i})^{p_{k_i}}| \ge [(f_{k_i}(x_{k_i} r))^{p_{k_i}} t_{k_i}]^{-1} \text{ for all } i \in N.$$

Then $[(f_{k_i}(\gamma_{k_i} x_{k_i} r))^{p_{k_i}} t_{k_i}] \ge 1$ for all $i \in N$.

Thus $(\gamma_k) \notin (m_0\{E_k, p\}, m_0\{E_k, p\})$, a contradiction.

Hence the result.

From Lemma 5 and Theorem 7, the following result follows.

Corollary 2. $M(E, E) = \ell_\infty$, *where* $E = m_0\{E_k, p\}$ *if and only if* $h = inf\ p_k > 0$ *and* $H = sup\ p_k < \infty$.

The following result follows from Lemma 5 and Corollary 2.

Corollary 3. *Let* $h = inf\ p_k$ *and* $H = sup\ p_k$. *Then the following are equivalent:*

(i) $H < \infty$ *and* $h > 0$

(ii) $m_0\{E_k, \Lambda, p\} = m_0(E_k, \Lambda, p)$

Theorem 7. *Let p, q* $\in \ell_\infty$. *Then* $\bar{c}_0\{E_k,\Lambda,p\}\subseteq \bar{c}_0\{E_k,\Lambda,q\}$ *if and only if there exists a subset K of N with* $\delta(K)=1$ *such that*

$$\lim_M \limsup_{k\in K} q_k^{-1}(M^{-1}p_k)^{\frac{q_k}{p_k}} = 0.\text{--------}(1)$$

Proof. Let $K\subset N$ be fixed with $\delta(K)=1$.

Let $I(M) = \limsup_{k\in K} q_k^{-1}(M^{-1}p_k)^{\frac{q_k}{p_k}}$, for all $M>1$

and $I(M,k) = q_k^{-1}(M^{-1}p_k)^{\frac{q_k}{p_k}}$.

Suppose (1) holds . Then for a given $\varepsilon>0$, there exists $M_0>1$ such that

$I(M)<\varepsilon$, for all $M>M_0$.

Let $x\in \bar{c}_0\{E_k,\Lambda,p\}$ and M^* be fixed with $M^*>M_0$. Then there exists $r>0$ such that

$(f_k(\gamma_k x_k r))^{p_k} t_k \xrightarrow{stat} 0$ as $k\to\infty$.

Then $\delta(A)=\delta(\{k\in N: (f_k(\gamma_k x_k r))^{p_k} t_k < M^{*-1}\})=1$.

Let $B=A\cap K$. Then $\delta(B)=1$. Now, for all $k\in B$,

$(f_k(\gamma_k x_k r))^{q_k} q_k^{-1} = q_k^{-1}(M^{*-1}p_k)^{\frac{q_k}{p_k}} \le \limsup_{k\in B} q_k^{-1}(M^{*-1}p_k)^{\frac{q_k}{p_k}} < \mathrm{I}(M^*)<\varepsilon$.

Therefore $x\in \bar{c}_0\{E_k,\Lambda,q\}$.

Hence $\bar{c}_0\{E_k,\Lambda,p\}\subseteq \bar{c}_0\{E_k,\Lambda,q\}$.

Conversely suppose that $\bar{c}_0\{E_k,\Lambda,p\}\subseteq \bar{c}_0\{E_k,\Lambda,q\}$ but (1) fails. Then we have two cases:

(*i*) For all subsets *K* of *N* with $\delta(K)=1$, we have $I(M)=\infty$ for every integer $M>1$.

(*ii*) For any subsets *K* of *N* with $\delta(K)=1$, $I(M)<\infty$ for all $M>1$ and $\lim_M I(M)>0$.

Case (*i*) There exists a strictly increasing sequence (k_i) of positive integers with $\delta(\{k_i: i\in N\})\neq 0$ such that

$I(i+1, k_i) > i$, $i=1,2,3$, ------.

Define (x_k) as follows :

$$x_k = \frac{(i+1)^{-t_k} p_k^{t_k}}{|\gamma_k|} I_k,\quad k=k_i,\ i\in N,$$

$=\theta_k$, therwise.

So, $(f_k(\gamma_k x_k 1))^{p_k} t_k = (i+1)^{-1}$, for all $k=k_i$, $i=1,2,3$, ------.

Hence $(x_k)\in \bar{c}_0\{E_k,\Lambda,p\}$.

Also, $(f_k(\gamma_k x_k r))^{q_k} q_k^{-1} =$

$r^{q_k} q_k^{-1} p_k^{\frac{q_k}{p_k}} (i+1)^{-\frac{q_k}{p_k}} \ge I(i+1, k_i)$ min (1, $r^{H'}$), for $k=k_i$ and for all $r>0$

$(H'=\sup_k q_k<\infty) > i$ min(1, $r^{H'}$)

Therefore $(x_k)\notin \bar{c}_0\{E_k,\Lambda,q\}$. Hence we arrive at a contradiction.

Case (*ii*) Suppose $\lim_M I(M) = 2a > 0$. Then there exists a strictly increasing sequence (k_i) of positive integers such that

$I(M+i-1, k_i) > a$, $i=1,2,3$, ------.

We define a sequence $x=(x_k)$ as follows :

$$x_k = \frac{(M+i-1)^{-t_k} p_k^{t_k}}{|\gamma_k|} I_k,\ k=k_i,\ i\in N, =\theta_k$$

,otherwise.

Then $(f_k(\gamma_k x_k 1))^{p_k} t_k = (M+i-1)^{-1}$, for all $k=k_i$, $i=1,2,3$, ------.

Hence $(x_k)\in \bar{c}_0\{E_k,\Lambda,p\}$.

Further for $k=k_i$, $i\in N$, we have

$(f_k(\gamma_k x_k r))^{q_k} q_k^{-1} =$

$r^{q_k} q_k^{-1} p_k^{\frac{q_k}{p_k}} (M+i-1)^{-\frac{q_k}{p_k}}$

$\geq I(M+i-1,\ k_i)\min(1,\ r^{H'})$,

for all $r > 0$

$(H' = \sup_k q_k < \infty) > a\ \min(1,\ r^{H'})$

Therefore $(x_k) \notin \bar{c}_0\{E_k, \Lambda, q\}$ at a contradiction.

Hence $\lim_M$ I (M) = 0.

This completes the proof of the theorem.

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