On A Class Of Statistically Null Vector Valued Sequences Associated With Multilpier Sequences

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Abstract : In this article we introduce the multiplier vector valued sequence space \( \tilde{c}_0 \{E_k, \Lambda, p\} \), where \( \Lambda = (\gamma_k) \) is an associated multiplier sequence of non-zero complex numbers and the terms of the sequence are chosen from the semi normed spaces \( E_k, k \in \mathbb{N} \). We study some properties of these spaces like completeness, solidity, inclusion of two different classes. We prove the decomposition theorem.

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1. Introduction

The notion of statistical convergence of sequences was introduced by Fast[2] and Schoenberg [13]. It is also found in zygmund[17]. Later on it was studied from sequence space point of view and linked with summability by Fridy [3], Connor [1], Šalát [12], Tripathy [15] and many others. The idea depends on the density of subsets of the set \( N \) of natural numbers. A subset \( E \) of \( N \) is said to have density \( \delta(E) \) if

\[
\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=n}^{\infty} \chi_E(k),
\]

where \( \chi_E \) is the characteristic function of \( E \).

For \( (x_k) \) and \( (y_k) \) two sequences, we say that \( x_k = y_k \) for almost all \( k \) (in short \( a.a.k \)) if \( \delta \{ k \in \mathbb{N} : x_k \neq y_k \} = 0 \).

A sequence \( (x_k) \) is said to be statistically convergent to \( L \) if for any \( \varepsilon > 0 \), \( \delta \{ k \in \mathbb{N} : |x_k - L| \geq \varepsilon \} = 0 \) and we write \( \text{stat-lim} \ x_k = L \).

A sequence \( (x_k) \) is said to be statistically Cauchy provided that for every \( \varepsilon > 0 \), there exists a number \( n_0 \) such that \( |x_k - x_{n_0}| < \varepsilon \) for \( a.a.k \).

The scope for the studies on sequences spaces was further extended by using the notion of associated multiplier sequences. Goes and Goes [4] defined the differentiated sequence space \( dE \) and the integrated sequence space \( \int E \) for a given sequence space \( E \), using multiplier sequences \( (\gamma_k) \) and \( (k!) \) respectively. Kamthan [5] used the multiplier sequence \( (k!) \). In the present article we shall consider a general multiplier sequence \( \Lambda = (\gamma_k) \) of non-zero scalars.

Throughout \( \ell_\infty, c, c_0 \) and \( \widetilde{c}_0 \) denote the spaces of bounded, convergent, null and statistically null sequences respectively.

2. Definitions And Preliminaries

The concept of paranormed sequence space was studied by Nakano [10] and Simons [14] at the initial stage. Later on it was studied by Maddox [9], Lascarides [7], and many others.
The studies on vector valued sequence spaces was explored by Ratha and Srivastava [11], Leonard [8] and many others.

Throughout the article $E_k$ will denote a seminormed sequence space, seminormed by $f_{k}$ for all $k \in N$, defined over $C$, the field of complex numbers. Throughout $p = (p_k)$ is a sequence of strictly positive real numbers and $t_k = p_k^{-1}$, for all $k \in N$.

A vector valued sequence space $E$ is said to be solid (or normal) if $\alpha x = (\alpha_k x_k) \in E$, whenever $x = (x_k) \in E$ and $\alpha = (\alpha_k)$ is a sequence of scalars such that $|\alpha_k| \leq 1$, for all $k \in N$.

A sequence space $E$ is said to be monotone if $E$ contains the canonical preimages of all its subspaces (one may refer to Kamthan and Gupta [6], p. 48).

A sequence space $E$ is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where $\pi$ is a permutation of $N$.

Tripathy and Sen [16] have studied the following vector valued sequence spaces:

- $\ell_\infty(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k \text{ for all } k \in N \}$ and $\sup (f_k(y_k x_k))^{p_k} < \infty$
- $c_0(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k \text{ for all } k \in N \}$ and $(f_k(y_k x_k))^{p_k} \to 0, \text{ as } k \to \infty$
- $\ell_\infty(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k \text{ for all } k \in N \}$ and there exists $r > 0$ such that $\sup(f_k(y_k x_k))^{p_k} t_k < \infty$
- $c_0(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k \text{ for all } k \in N \}$ and there exists $r > 0$ such that $(f_k(y_k x_k))^{p_k} t_k \to 0, \text{ as } k \to \infty$

If $p \in \ell_\infty$ and each $E_k$ is complete, then $c_0(E_k, \Lambda, p)$ is a complete paranormed space, paranormed by $g(\alpha) = \sup_k (f_k(y_k x_k p^{-k})^{p_k} t_k)\sup_k (f_k(y_k x_k p^{-k})^{p_k} t_k)$, where $M = \max(1, H), H = \sup_k p_k$.

The space $\ell_\infty(E_k, \Lambda, p)$ is paranormed by $g$ if $\inf p_k > 0$.

In this article we introduce the following class of vector valued sequence spaces associated with multiplier sequences:

- $c_0(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k \text{ for all } k \in N \}$ and $(f_k(y_k x_k))^{p_k} t_k \to 0, \text{ as } k \to \infty$
- $c_0(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k \text{ for all } k \in N \}$ and there exists $r > 0$ such that $(f_k(y_k x_k r))^{p_k} t_k \to 0, \text{ as } k \to \infty$.

Let $\delta_0(E_k) = \{ (x_k) : x_k \in E_k \text{ for all } k \in N \}$ and $\delta_0(E_k) = \{ (x_k) : x_k = \theta_k \}$, where $\theta_k$ is the zero element of $E_k$.

Two sequence spaces $E$ and $F$ are said to be equivalent if there exists a sequence $u = (u_k)$ of strictly positive numbers such that the mapping $u : E \to F$ defined by $y = u x = (u_k x_k) \in F$, whenever $(x_k) \in E$, is a one-to-one correspondence between $E$ and $F$. It is denoted by $E \cong F$ or simply $E \cong F$ (see for instance Nakano [10]).

It is remarked by Lascarides ([7] Remark 3) that “If $E$ is a sequence space paranormed (or normed) by $g$ and $E \cong F(u)$, then $F$ is a sequence space paranormed (or normed) by $g_u$ defined by $g_u(y) = g(u^y), y \in F$.”

Further it is noted by Lascarides [7] that “if $(p_1) \in \ell_\infty^\prime$ and $c_0(p) \cong c_0(p)(u)$, then $\ell_\infty(p) \cong \ell_\infty(p)(u)$, where $u = (p_k^u)$.”

For $E, F$ two sequence spaces we define $M(F, E)$ as follows:
The following statements are equivalent

Moreover, for all $(x_k) \in F$, we have

**Lemma 1.** A sequence space $E$ is solid implies that $E$ is monotone.

**Lemma 2.** (Frídy [3], Theorem 1). The following statements are equivalent:

(i) $(x_k)$ is a statistically convergent sequence
(ii) $(x_k)$ is statistically Cauchy sequence
(iii) $(x_k)$ is a sequence for which there is a convergent sequence $(y_k)$ such that $x_k = y_k$ for a.a.k

**Lemma 3.** (Šalát [12], Lemma 1.1). A sequence $(x_k)$ statistically converges to L if and only if there exists such a set $K = \{k_1 < k_2 < k_3 < \ldots \} \subset N$ that $\delta(K) = 1$ and $\lim_{n \to \infty} x_{k_n} = L$.

**Lemma 4.** (Connor [1], Theorem 2.3). If $x = (x_k)$ is statistically convergent to L, then there is a convergent sequence $y = (y_k)$ and a statistically null sequence $z = (z_k)$ such that $y$ is convergent to $L$, $x = y + z$ and $\delta(\{ k \in N : z_k \neq 0 \}) = 0$. Moreover, if $x$ is bounded then $z$ is bounded and $\|z\|_\infty \leq \|x\|_\infty + \|L\|$.

**Lemma 5.** (Lascarides [7], Proposition 1). Let $h = \inf p_k$ and $H = \sup p_k$. Then the following conditions are equivalent:

(i) $H < \infty$ and $h > 0$
(ii) $c_0(p) = c_0$ or $\ell_\infty(p) = \ell_\infty$
(iii) $\ell_\infty(p) = \ell_\infty(p)$
(iv) $c_0\{p\} = c_0(p)$
(v) $\ell\{p\} = \ell(p)$.

3. Main Results

In this section we prove the results of this article.

**Theorem 1.** $c_0\{E_k, \Lambda, p\}$ is a linear space for any sequence $p = (p_k)$.

**Proof.** Let $x \in c_0\{E_k, \Lambda, p\}$. Then there exists $r > 0$ such that $(f_k(y_k x_k r)) p_k t_k \rightarrow 0$, as $k \to \infty$.

Let $\xi \in C$ and without loss of generality let $\xi \neq 0$. Let $\rho = r \mid \xi \mid^l > 0$, then we have $(f_k(y_k (\xi x_k)(\rho))) p_k t_k = (f_k(y_k x_k r)) p_k t_k \rightarrow 0$, as $k \to \infty$.

Therefore $\xi x \in c_0\{E_k, \Lambda, p\}$, for all $\xi \in C$ and $x \in c_0\{E_k, \Lambda, p\}$.

Next we suppose that $x, y \in c_0\{E_k, \Lambda, p\}$. Then there exists $r_1 > 0$ and $r_2 > 0$ such that $(f_k(y_k x_k r_1)) p_k t_k \rightarrow 0$ as $k \to \infty$ and $(f_k(y_k x_k r_2)) p_k t_k \rightarrow 0$ as $k \to \infty$.

Then for a given $\varepsilon > 0$, we have $\delta(K_1) = \delta(\{ k \in N : (f_k(y_k x_k)(\rho)) p_k t_k < \varepsilon \}) = 1$ and $\delta(K_2) = \delta(\{ k \in N : (f_k(y_k x_k)(\rho)) p_k t_k < \varepsilon \}) = 1$.

Let $r = r_1 r_2 (r_1 + r_2)^{-1}$ and $K = K_1 \cap K_2$. Then clearly $\delta(K) = 0$ and for all $k \in K$, we have $(f_k(y_k x_k r)) p_k t_k \leq [(f_k(r_1 y_k x_k r_2 r_1 + r_2)^{-1}) p_k t_k + (f_k(r_2 y_k x_k r_1 r_1 + r_2)^{-1}) p_k t_k < \varepsilon \ p_k$.

Hence $x + y \in c_0\{E_k, \Lambda, p\}$.

Since $\ell_\infty\{E_k, \Lambda, p\}$ is a linear space for any $p$, we have the following result.

**Corollary 1.** $m_0\{E_k, \Lambda, p\}$ is a linear space for any sequence $p = (p_k)$.

The proofs of the following two results are easy, so omitted.
Theorem 2. Let \(0 < \inf p_k \leq p_k \leq \sup p_k\) and each \(E_k\) is complete, then \(m_0 \{E_k, \Lambda, p\}\) is a complete paranormed space, paranormed by \(g\).

Proposition 3. The space \(m_0 \{E_k, \Lambda, p\}\) is solid as well as monotone.

Proposition 4. The space \(m_0 \{E_k, \Lambda, p\}\) is not symmetric.

Proof. The result follows from the following example.

Let \(E_k = c\), \(p_k = 1\) and \(\gamma_k = 1\), for all \(k \in N\).

Also let \(f_k(x_k) = \sup_i |x'_k|\),

where \(x_k = (x'_k)\), for all \(k \in N\).

We consider the sequence \((x_k)\) defined by \(x_k = e\), if \(k = i^2\), \(i \in N\),

\[= \bar{\delta}, \text{ otherwise.}\]

where \(e = (1, 1, \ldots)\) and \(\bar{\delta} = (0, 0, \ldots)\).

Then \((x_k) \in m_0 \{E_k, \Lambda, p\}\).

We consider the rearrangement \((y_k)\) of \((x_k)\) as \(y_k = e\), if \(k\) is odd

\[= \bar{\delta}, \text{ otherwise.}\]

Then \((y_k) \notin m_0 \{E_k, \Lambda, p\}\). Hence \(m_0 \{E_k, \Lambda, p\}\) is not symmetric.

Theorem 5. If \(p \in \ell_\infty\), then the following are equivalent

(i) \((x_k) \in \overline{c}_0 \{E_k, \Lambda, p\}\.

(ii) there exists a subset \(K = \{k_1, k_2, k_3, \ldots\}\) of \(N\) such that \(\delta(K) = 1\) and

\[\lim_{i \to \infty} \left( (f_k(\gamma_k x_k r))^{p_i} t_k \right) = 0.\]

(iii) there exists \((y_k) \in c_0 \{E_k, \Lambda, p\}\) such that \(x_k = y_k\) for a.a.k.

(iv) there exists sequences \((y_k)\) and \((z_k)\) such that \(x_k = y_k + z_k\) for all \(k \in N\) and

\[(y_k) \in c_0 \{E_k, \Lambda, p\}, \ (z_k) \in \delta_0 (E_k).\]

Proof. Let \((x_k) \in \overline{c}_0 \{E_k, \Lambda, p\}\). Then there exists \(r > 0\) such that

\[(f_k(\gamma_k x_k r))^{p_i} t_k \xrightarrow{\text{stat}} 0\text{ as } k \to \infty.\]

Let us consider \(a_k = (f_k(\gamma_k x_k r))^{p_i} t_k\). Then \((a_k) \in \overline{c}_0\).

The equivalence of (i) and (ii) follows from lemma 3, that of (i) and (iii) from lemma 2 and that of (i) and (iv) from lemma 4.

Theorem 6. Let \((p_k)\) be a given sequence of strictly positive numbers. Then \((\gamma_k) \in (E, E)\) if and only if \((\gamma_k)^{p_k}) \in \ell_\infty\), where \(E = m_0 \{E_k, p\}\).

Proof. The sufficiency is obvious.

For the necessity, suppose that \((\gamma_k)^{p_k}) \notin \ell_\infty\).

Then there exists a subsequence \((\gamma_k)^{p_k})\) of \((\gamma_k)^{p_k})\) such that

\[(\gamma_k)^{p_k}) \to \infty, \text{ as } i \to \infty.\]

Then we can find a sequence \((x_k) \in m_0 \{E_k, p\}\) such that

\[|(\gamma_k)^{p_k})| \geq \left[ (f_k(\gamma_k x_k r))^{p_i} t_k \right]^{-1} \text{ for all } i \in N.\]

Then \([(f_k(\gamma_k x_k r))^{p_i} t_k] \geq 1\) for all \(i \in N\).

Thus \((\gamma_k) \notin (m_0 \{E_k, p\}, m_0 \{E_k, p\})\), a contradiction.

Hence the result.

From Lemma 5 and Theorem 7, the following result follows.

Corollary 2. \(M(E, E) = \ell_\infty\), where \(E = m_0 \{E_k, p\}\) if and only if

\[h = \inf p_k > 0 \text{ and } H = \sup p_k < \infty.\]

The following result follows from Lemma 5 and Corollary 2.

Corollary 3. Let \(h = \inf p_k\) and \(H = \sup p_k\). Then the following are equivalent:

(i) \(H < \infty\) and \(h > 0\)

(ii) \(m_0 \{E_k, \Lambda, p\} = m_0 \{E_k, \Lambda, p\}\)
Theorem 7. Let \( p, q \in \ell_{\infty} \). Then
\[
\overline{c_0} \{ E_k, \Lambda, p \} \subseteq \overline{c_0} \{ E_k, \Lambda, q \} \quad \text{if and only if there exists a subset } K \text{ of } N \text{ with } |\delta(K)| = 1 \text{ such that}
\]
\[
\lim_{M \to \infty} \sup_{k \in K} q_k^{-1} \left( M^{-1} p_k \right)^{\frac{q}{p}} = 0. \tag{1}
\]
Proof. Let \( K \subset N \) be fixed with \( |\delta(K)| = 1 \).

Let \( I(M) = \limsup_{k \in K} q_k^{-1} \left( M^{-1} p_k \right)^{\frac{q}{p}} \), for all \( M > 1 \)
and \( I(M, k) = q_k^{-1} \left( M^{-1} p_k \right)^{\frac{q}{p}} \).

Suppose (1) holds. Then for a given \( \varepsilon > 0 \), there exists \( M_0 > 1 \) such that
\[
I(M) < \varepsilon, \quad \text{for all } M > M_0.
\]
Let \( x \in \overline{c_0} \{ E_k, \Lambda, p \} \) and \( M^* > M_0 \). Then there exists \( r > 0 \) such that
\[
(f_k(\gamma_k x_k r))^{\underline{A}} t_k \xrightarrow{\text{st}} 0 \text{ as } k \to \infty.
\]
Then \( \delta(A) = \delta \{ k \in N : (f_k(\gamma_k x_k r))^{\underline{A}} t_k < M^{*-1} \} = 1 \).

Let \( B = A \cap K \). Then \( \delta(B) = 1 \). Now, for all \( k \in B \),
\[
(f_k(\gamma_k x_k r))^{\underline{A}} d_k^{-1} = q_k^{-1} \left( M^{*-1} p_k \right)^{\frac{q}{p}} \leq I(M^*).
\]

Therefore \( x \in \overline{c_0} \{ E_k, \Lambda, p \} \).
Hence \( \overline{c_0} \{ E_k, \Lambda, p \} \subseteq \overline{c_0} \{ E_k, \Lambda, q \} \).

Conversely suppose that
\[
\overline{c_0} \{ E_k, \Lambda, p \} \subseteq \overline{c_0} \{ E_k, \Lambda, q \} \quad \text{but (1) fails}. \text{Then we have two cases}:
\]
(i) For all subsets \( K \subset N \) with \( |\delta(K)| = 1 \), we have \( I(M) = \infty \) for every integer \( M > 1 \).
(ii) For any subsets \( K \subset N \) with \( |\delta(K)| = 1 \), \( I(M) < \infty \) for all \( M > 1 \) and \( \lim_{M \to \infty} I(M) = 0 \).

Case (i) There exists a strictly increasing sequence \( (k_i) \) of positive integers with \( \delta \{ k_i : i \in N \} \neq 0 \) such that
\[
\lim I(i + 1, k_i) > i, \quad i = 1, 2, 3, \ldots,
\]
Define \( (x_k) \) as follows:
\[
x_k = \frac{(i + 1)^{\frac{q_i}{p_i}} t_k}{|\gamma_k|}, \quad k = k_i, i \in N,
\]
otherwise.

So, \( (f_k(\gamma_k x_k 1))^{\underline{A}} t_k = (i + 1)^{-1} \), for all \( k = k_i, i = 1, 2, 3, \ldots \).

Hence \( (x_k) \in \overline{c_0} \{ E_k, \Lambda, p \} \). Also,
\[
(f_k(\gamma_k x_k r))^{\underline{A}} q_k^{-1} = r^{q_i} q_k^{-1} p_k^{\frac{q_i}{p_i}} (i + 1)^{\frac{q_i}{p_i}} \geq I(i + 1, k_i)
\]
min \( (1, r^{q_i} ) \), for \( k = k_i \) and for all \( r > 0 \)
\[
(H' = \sup q_k < \infty) \Rightarrow i \min(1, r^{q_i} )
\]
Therefore \( (x_k) \in \overline{c_0} \{ E_k, \Lambda, q \} \). Hence we arrive at a contradiction.

Case (ii) Suppose \( \lim_{M \to \infty} I(M) = 2a > 0 \). Then there exists a strictly increasing sequence \( (k_i) \) of positive integers such that
\[
I(M + i - 1, k_i) > a, \quad i = 1, 2, 3, \ldots
\]
We define a sequence \( x = (x_k) \) as follows:
\[
x_k = \frac{(M + i - 1)^{\frac{q_i}{p_i}} t_k}{|\gamma_k|}, \quad k = k_i, i \in N, \quad \theta_k
\]
otherwise.

Then \( (f_k(\gamma_k x_k 1))^{\underline{A}} t_k = (M + i - 1)^{-1} \), for all \( k = k_i, i = 1, 2, 3, \ldots \).

Hence \( (x_k) \in \overline{c_0} \{ E_k, \Lambda, p \} \).
Further for \( k = k_i, i \in N \), we have
\[
(f_k(\gamma_k x_k r))^{\underline{A}} q_k^{-1} =
\]
\[
= \frac{r^{q_i} q_k^{-1} p_k^{\frac{q_i}{p_i}} (M + i - 1)^{\frac{q_i}{p_i}}}{|\gamma_k|}
\]

\[ \geq I(\mathcal{M} + i - 1, k_i) \min_1(1, r^{\mathcal{H}}), \]

for all \( r > 0 \).

\( (\mathcal{H} = \sup_{k} q_k < \infty) \geq a \min(1, r^{\mathcal{H}}) \)

Therefore \( (x_k) \notin \tilde{c}_0 \{E_k, \Lambda, q\} \) at a contradiction.

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Hence \( \lim_{\mathcal{M}} I(\mathcal{M}) = 0. \)

This completes the proof of the theorem.

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