Turkish Journal of Science & Technology
Volume 4, No 1, 45-56, 2009

Exact and Numerical Solutions of the Fifth Order KdV Equations and Couple KdV System by Using to Direct Algebraic Method

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(Received: 19.11.2008; Accepted: 24.03.2009)

Abstract

In this paper, we implemented a direct algebraic method for the exact solutions of the fifth order Korteweg-de Vries (KdV) equations and couple KdV system. By using this scheme, we found several exact solutions of the three fifth order KdV equations and one couple KdV system. In this work we consider how Adomian's decomposition method (ADM), homotopy analysis method (HAM) can be used to investigate propagating traveling solitary wave solutions of fifth order KdV equation. It is also worth noting that the advantage of the approximation of the series methodologies displays a fast convergence of the solutions. The illustrations show the dependence of the rapid convergence depends on the character and behavior of the solutions just as in a closed form solutions.

Key Words: The Korteweg-de Vries Equation; Couple KdV System; Direct Algebraic Method; Adomian Decomposition Method; Homotopy Analysis Method.

Doğrudan Cebirsel Metodu Kullanarak Beşinci Mertebeden KdV Denklemleri ve Çift KdV Sisteminin Tam ve Sayısal Çözümleri

Özet

Bu çalışmada beşinci mertebeden KdV denklemleri ve çift KdV sisteminin tam çözümleri için doğrudan cebirsel metodu sunacağız. Bu tekniği kullanarak beşinci mertebeden KdV denklemleri ve çift KdV sisteminin birkaç tane tam çözümünü bulacağız. Ayrıca bu çalışmada beşinci mertebeden KdV denkleminin hareket eden dalga çözümleri için Adomian ayrışım metodunu ve Homotopy analiz metodunu kullanacağız. Bu seri metotlarının avantajı çözümlere hızlı bir yakınsama göstermesidir. Açıklamalar kapalı formdaki çözümlerin davranışlarına ve karakterine bağlı olarak hızlı bir yakınsama olduğunu gösterir.

Anahtar Kelimeler: KdV Denklemi; Çift KdV Sistemi; Doğrudan Cebirsel Metot; Adomian Ayrışım Metot; Homotopy Analiz Metot.

1. Introduction

The theory of nonlinear dispersive wave motion has recently undergone much study. We do not attempt to characterize the general form of nonlinear dispersive wave equations [1, 2]. Nonlinear phenomena play a crucial role in applied mathematics and physics. Furthermore, when an original nonlinear equation is directly calculated, the solution will preserve the actual physical characters of solutions [3]. Explicit solutions to the nonlinear equations are of fundamental importance. Various methods for obtaining explicit solutions to nonlinear evolution equations have been proposed. Many explicit exact methods have been introduced in literature [4, 17]. Among them are Generalized Miura Transformation, Darboux Transformation, Cole-Hopf Transformation, Hirota's dependent variable Transformation, the inverse scattering Transform and the Bäcklund Transformation, tanh method, sine-cosine method, Painleve method, homogeneous balance method, similarity reduction method, improved tanh method and so on. In fact, recently a direct algebraic approach has been constructed an automated tanh-function method by Parkes and Duffy [12]. The authors present a *Mathematica* package that deals with complicated algebraic and outputs directly the required solutions for particular nonlinear equations. In this study, we implemented a new method for finding the exact solutions of the fifth order KdV equation and a couple KdV systems.

2. An Analysis of the Method and Applications

Before starting to give a detail of the method, we will give a simple description of the direct algebraic method [18]. For doing this, one can consider in a two independent variables general form of nonlinear PDE

$$Q(u,u_t,u_x,u_{xx},\ldots)=0, \tag{1}$$

and transform Eq. (1) with

$u(x,t)=u(\xi),\ \xi=kx+wt,$

where k and w are arbitrary constants. After the transformation, we get a nonlinear ODE for $u(\xi)$

$$Q'(u',u'',u''',\ldots)=0. \tag{2}$$

Then, the solution of the Eq. (2) we are looking for is expressed in the form of

$$u(x,t)=u(\xi)=\sum_{i=0}^{M}a_iF^i, \tag{3}$$

where M is a positive integer that can be determined by balancing the highest order derivative with the highest nonlinear terms in equation, and the coefficients of series $k,w,a_0,a_1,\ldots,a_M$ are parameters to be determined. Substituting solution (3) into Eq. (2) yields a set of algebraic equations for $F'F^i$ or F^i, then, all coefficients of $F'F^i$ or F^i has to vanish. After this separated algebraic equation, we can found $k,w,a_0,a_1,\ldots,a_M$ constants.

In this work, we aim to solve the shallow water wave equation by using the direct algebraic method which is introduced by Zhang [18].

$$F'^2=q_3F^3+q_2F^2, \tag{4}$$

where $F'=\dfrac{dF}{d\xi}$ and q_2, q_3 are constants.

Example 1. Consider a simple fifth order KdV equation [19]

$$u_t+uu_x+u_{xxxxx}=0. \tag{5}$$

Let us consider the traveling wave solutions $u(x,t)=u(\xi),\ \ \xi=kx+wt$, then Eq. (5) becomes

$$wu'+kuu'+k^5u^{(5)}=0. \tag{6}$$

Balancing uu' with $u^{(5)}$ gives *M=2.* Therefore, we may choose

$$u=a_0+a_1F+a_2F^2. \tag{7}$$

Substituting (7) into Eq. (6) yields a set of algebraic equations for a_0,a_1,a_2,k,w. These equations are found as

$$\begin{aligned}
&a_0a_1k+a_1k^5q_2^2+a_1w=0,\\
&a_1^2k+2a_0a_2k+32a_2k^5q_2^2+15a_1k^5q_2q_3+2a_2w=0,\\
&3a_1a_2k+195a_2k^5q_2q_3+\frac{45}{2}a_1k^5q_3^2=0,\\
&2a_2^2k+210q_2k^5q_3^2=0,
\end{aligned} \tag{8}$$

From the solution of the system, we can be found

$$\begin{aligned}
&a_0=a_0,\ a_1=0,\ a_2=-105k^4q_3^2\\
&w=-a_0k,\ k=k,\ q_2=0
\end{aligned} \tag{9}$$

with the aid of *Mathematica*. Substituting (7) and (9) into (4) we have obtained the following solution of Eq. (5).

$$u(x,t)=a_0-\frac{105k^4q_3^2}{\left(\frac{q_3}{4}(kx-a_0kt)^2+c_1(kx-a_0kt)+c_2\right)^2} \tag{10}$$

where $c_1^2=q_3c_2$. One can find the constant $q_2=0$ in the algebraic equation (8) so that we obtained only one solution for Eq. (5). In the following example, we have got three different types of solutions for Eq. (11).

Example 2. Consider the fifth order KdV equation [19]

$$u_t+uu_x-uu_{xxx}+u_{xxxxx}=0. \tag{11}$$

Let us consider the traveling wave solutions $u(x,t)=u(\xi),\ \xi=kx+wt$ then Eq. (11) becomes

$$wu'+kuu'-k^3uu'''+k^5u^{(5)}=0. \tag{12}$$

Balancing uu''' with $u^{(5)}$ gives *M=1.* Therefore, we may choose

$$u=a_0+a_1F. \tag{13}$$

Substituting (13) into Eq. (12) yields a set of algebraic equations for a_0, a_1, k, w. These equations are found as

$$-a_0 a_1 k^3 q_2 + a_1 k^5 q_2^2 + a_0 a_1 k + a_1 w = 0,$$
$$-a_1^2 k^3 q_2 - 3a_0 a_1 k^3 q_3 + 15 a_1 k^5 q_2 q_3 + a_1^2 k = 0,$$
$$-3a_1^2 k^3 q_3 + \frac{45}{2} a_1 k^5 q_3^2 = 0. \tag{14}$$

From the solutions of the system, we can obtain

$$a_0 = \frac{5}{2}\left(1 + k^2 q_2\right),\ a_1 = \frac{15k^2 q_3}{2},\quad w = \frac{1}{2}k\left(-5 + 3k^4 q_2^2\right),\ k \neq 0,\ q_3 \neq 0 \tag{15}$$

with the aid of *Mathematica*. Substituting (13) and (15) into (4) we have obtained the following solutions of Eq. (11). These solutions are:

Solution 1:

$$u(x,t) = \frac{5}{2}\left(1 + k^2 q_2\right) + \frac{\frac{15k^2 q_3}{2}}{c_1 e^{\sqrt{q_2}\left(kx + \frac{1}{2}k\left(-5+3k^4 q_2^2\right) t\right)} + c_2 e^{-\sqrt{q_2}\left(kx + \frac{1}{2}k\left(-5+3k^4 q_2^2\right) t\right)} - \frac{q_3}{2q_2}}, \tag{16}$$

where $q_2 > 0,\ 16 c_1 c_2 q_2^2 = q_3^2$.

Solution 2:

$$u(x,t) = \frac{5}{2}\left(1 + k^2 q_2\right) + \frac{\frac{15k^2 q_3}{2}}{c_1 Cos\left(\sqrt{-q_2}\,\xi\right) + c_2 Sin\left(\sqrt{-q_2}\,\xi\right) - \frac{q_3}{2q_2}}, \tag{17}$$

where $q_2 < 0,\ 4q_2^2\left(c_1^2 + c_2^2\right) = q_3^2,\ \xi = kx + \frac{1}{2}k\left(-5 + 3k^4 q_2^2\right)t$.

Solution 3:

$$u(x,t) = \frac{5}{2} + \frac{\frac{15k^2 q_3}{2}}{\frac{q_3}{4}\left(kx + \frac{1}{2}k\left(-5+3k^4 q_2^2\right) t\right)^2 + c_1\left(kx + \frac{1}{2}k\left(-5+3k^4 q_2^2\right) t\right) + c_2}, \tag{18}$$

where $c_1^2 = q_3 c_2$ when $q_2 = 0$.

Example 3. Consider the fifth order KdV equation [19]

$$u_t + u_x + u u_x + u_{xxx} + u u_{xxx} + u_{xxxxx} = 0. \tag{19}$$

Let us consider the traveling wave solutions $u(x,t) = u(\xi),\ \xi = kx + wt$ then Eq. (19) becomes

$$(w+k)u' + kuu' + k^3 u''' + k^3 u u''' + k^5 u^{(5)} = 0. \tag{20}$$

Balancing uu''' with $u^{(5)}$ gives *M=1*. Therefore, we may choose

$$u = a_0 + a_1 F. \tag{21}$$

Substituting (21) into Eq. (20) yields a set of algebraic equations for a_0, a_1, k, w. These equations are found as

$$\begin{aligned} & a_1 k^3 q_2 + a_0 a_1 k^3 q_2 + a_1 k^5 q_2^2 + a_1 k + \\ & + a_0 a_1 k + a_1 w = 0 \\ & a_1^2 k^3 q_2 + 3a_1 k^3 q_3 + 3a_0 a_1 k^3 q_3 + \\ & + 15 a_1 k^5 q_2 q_3 + a_1^2 k = 0, \\ & 3a_1^2 k^3 q_3 + \frac{45}{2} a_1 k^5 q_3^2 = 0, \end{aligned} \tag{22}$$

From the solutions of the system, we can be found

$$a_0 = \frac{1}{2}\left(3 - 5k^2 q_2\right),\ a_1 = -\frac{15k^2 q_3}{2}$$
$$w = \frac{k}{2}\left(-5 + 3k^4 q_2^2\right),\ k \neq 0,\ q_3 \neq 0 \qquad (23)$$

with the aid of *Mathematica*. Substituting (21) and (23) into (4) we have obtained the following solutions of equation (19). These solutions are:

Solution 1:

$$u(x,t) = \frac{1}{2}\left(3 - 5k^2 q_2\right) - \frac{15k^2 q_3}{2}\left(\frac{1}{c_1 e^{\sqrt{q_2}\xi} + c_2 e^{-\sqrt{q_2}\xi} - \frac{q_3}{2q_2}}\right), \qquad (24)$$

where $q_2 > 0,\ 16c_1 c_2 q_2^2 = q_3^2,\ \xi = kx + \frac{k}{2}\left(-5 + 3k^4 q_2^2\right)$.

Solution 2:

$$u(x,t) = \frac{1}{2}\left(3 - 5k^2 q_2\right) - \frac{15k^2 q_3}{2}\left(\frac{1}{c_1 \cos\sqrt{-q_2}\,\xi + c_2 \sin\sqrt{-q_2}\,\xi - \frac{q_3}{2q_2}}\right), \qquad (25)$$

where $q_2 < 0,\ 4q_2^2\left(c_1^2 + c_2^2\right) = q_3^2,\ \xi = kx + \frac{k}{2}\left(-5 + 3k^4 q_2^2\right)$.

Solution 3:

$$u(x,t) = \frac{1}{2}\left(3 - 5k^2 q_2\right) - \frac{15k^2 q_3}{2}\left(\frac{1}{\frac{q_3}{4}\xi^2 + c_1\xi + c_2}\right), \qquad (26)$$

where $c_1^2 = q_3 c_2$ when $q_2 = 0$, $\xi = kx + \frac{k}{2}\left(-5 + 3k^4 q_2^2\right)$.

Example 4. For comparison reason we have considered following couple KdV system [20]

$$u_t + 6uu_x - 6vv_x + u_{xxx} = 0,$$
$$v_t + 3uv_x + v_{xxx} = 0. \qquad (27)$$

Let us take the transformations as $u(x,t) = u(\xi)$ and $v(x,t) = v(\xi),\ \xi = kx + wt$, then Eq. (27) becomes

$$wu' + 6kuu' - 6kvv' + u''' = 0,$$
$$wv' + 3kuv' + k^3 v''' = 0. \qquad (28)$$

The solutions of the systems, we are looking for is stated in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{M_1} a_i F^i,$$
$$v(x,t) = v(\xi) = \sum_{i=0}^{M_2} b_i F^i.$$

Balancing the highest linear terms with the highest nonlinear terms in Eq. (28) we can found $M_1 = 1,\ M_2 = 1$ and

$$u = a_0 + a_1 F,$$
$$v = b_0 + b_1 F. \qquad (29)$$

Substituting (29) into Eq. (28) yields a set of algebraic equations for k, w, a_0, a_1, b_0, b_1

$$6a_0 a_1 k - 6b_0 b_1 k + a_1 k^3 q_2 + a_1 w = 0,$$
$$6a_1^2 k - 6b_1^2 k + 3a_1 k^3 q_3 = 0,$$
$$3a_0 b_1 k + b_1 k^3 q_2 + b_1 w = 0, \qquad (30)$$
$$3a_1 b_1 k + 3b_1 k^3 q_3 = 0,$$

From the solutions of the system, we can be found

$$a_0 = a_0 \ , a_1 = -k^2 q_3 \ \ , b_0 = \mp \frac{a_0}{\sqrt{2}}$$

$$b_1 = \pm \frac{k^2 q_3}{\sqrt{2}} \ , w = -k\left(3a_0 + k^2 q_2\right) \quad (31)$$

$k \neq 0$, $q_3 \neq 0$,

with the aid of *Mathematica*. Substituting (29) and (31) into (4) we have obtained the solutions of the system of Eq. (27). These solutions could be getting in different cases

Case 1:

$$u(x,t) = a_0 - \frac{k^2 q_3}{c_1 e^{\sqrt{q_2}\,(kx - k(3a_0 + k^2 q_2)t)} + c_2 e^{-\sqrt{q_2}\,(kx - k(3a_0 + k^2 q_2)t)} - \frac{q_3}{2q_2}},$$

$$v(x,t) = \mp \frac{a_0}{\sqrt{2}} \pm \frac{\frac{k^2 q_3}{\sqrt{2}}}{c_1 e^{\sqrt{q_2}\,(kx - k(3a_0 + k^2 q_2)t)} + c_2 e^{-\sqrt{q_2}\,(kx - k(3a_0 + k^2 q_2)t)} - \frac{q_3}{2q_2}}, \quad (32)$$

where $q_2 > 0$, $16 c_1 c_2 q_2^2 = q_3^2$.

Case 2:

$$u(x,t) = a_0 - \frac{k^2 q_3}{c_1 Cos\left(\sqrt{-q_2}\,\xi\right) + c_2 Sin\left(\sqrt{-q_2}\,\xi\right) - \frac{q_3}{2q_2}}$$

$$v(x,t) = \mp \frac{a_0}{\sqrt{2}} \pm \frac{\frac{k^2 q_3}{\sqrt{2}}}{c_1 Cos\left(\sqrt{-q_2}\,\xi\right) + c_2 Sin\left(\sqrt{-q_2}\,\xi\right) - \frac{q_3}{2q_2}} \quad (33)$$

where $q_2 < 0$, $4q_2^2\left(c_1^2 + c_2^2\right) = q_3^2$, $\xi = kx - k\left(3a_0 + k^2 q_2\right)t$.

Case 3:

$$u(x,t) = a_0 - \frac{k^2 q_3}{\frac{q_3}{4}\left(kx - k\left(3a_0 + k^2 q_2\right)t\right)^2 + c_1\left(kx - k\left(3a_0 + k^2 q_2\right)t\right) + c_2},$$

$$v(x,t) = \mp \frac{a_0}{\sqrt{2}} \pm \frac{\frac{k^2 q_3}{\sqrt{2}}}{\frac{q_3}{4}\left(kx - k\left(3a_0 + k^2 q_2\right)t\right)^2 + c_1\left(kx - k\left(3a_0 + k^2 q_2\right)t\right) + c_2} \quad (34)$$

where $c_1^2 = q_3 c_2$ when $q_2 = 0$.

3. Numerical Methods and Their Application

i) Adomian Decomposition Method:

The aim of the present section is to give an outline and implement Adomian decomposition method (ADM) for nonlinear wave equations to obtain analytic and approximate solutions which are obtained in a rapidly convergent series with elegantly computable components by this method. The approach is based on the choice of a suitable differential operator which may be ordinary or partial, linear or nonlinear, deterministic or stochastic [8, 21-23]. It allows obtaining a decomposition series analytic solution of the equation which is calculated in the form of a convergent power series with easily computable components.

ADM is valid for ordinary and partial differential equations, no matter whether they contain small/large parameters, and thus is rather general. Moreover, the Adomian approximation series converge quickly. However, this method has some restrictions. Approximates solutions found by ADM often contain polynomials. In general, convergence regions of power series are small, thus acceleration techniques are often needed to enlarge convergence regions. This is mainly due to the fact that in generally power series is not an efficient set of base functions to approximate a nonlinear problem, but unfortunately ADM does not provide us with freedom to use different base functions.

We outlined of the method here in order to obtain analytic and approximate solutions using the ADM, consider the fifth order KdV equation [19]

$$L_t u + uu_x + u_{xxxxx} = 0, \tag{35a}$$

$$u(x,0) = g(x) = a_0 - \frac{105k^4q_3^2}{\left(\frac{q_3}{4}(kx)^2 + c_1(kx) + c_2\right)^2}, \tag{35b}$$

where $c_1^2 = q_3 c_2$, in an operator form of this equation can be written as

$$L_t(u) + (Nu) + u_{xxxxx} = 0, \tag{36}$$

where $L_t = \frac{\partial}{\partial t}$ and $Nu = uu_x$. It is assumed that L_t^{-1} is an integral operator given by $L_t^{-1} = \int_0^t (.)dt$.

Operating with the integral operator L_t^{-1} on both sides of (36) and we have

$$L_t^{-1}L_t(u) = -L_t^{-1}\left((Nu) + u_{xxxxx}\right). \tag{37}$$

Therefore, it follows that

$$u(x,t) = u(x,0) - L_t^{-1}\left((Nu) + u_{xxxxx}\right).$$

Let us find the zeroth component is obtained by

$$u_0 = u(x,0) \tag{38}$$

which is defined by all terms that arise from the initial condition and from integrating the source term and decompose the unknown function $u(x,t)$ a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{39}$$

The nonlinear term uu_x can be decomposed into the infinite series of polynomial given by

$$Nu = uu_x = \sum_{n=0}^{\infty} A_n,$$

where the components $u_i(x,t)$ will be determined recurrently, and A_n polynomials are the so-called Adomian polynomial [8, 24] of $u_0, u_1, u_2, \cdots, u_i$ defined by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\Phi\left(\sum_{k=1}^{\infty}\lambda^k u_k\right)\right]\Bigg|_{\lambda=0}, \quad n \geq 0. \tag{40}$$

Substituting (40) and (38) into (37) gives rise to

$$u_{n+1} = -L_t^{-1}\left(A_{n-1} + (u_n)_{xxxxx}\right), \quad n \geq 1 \tag{41}$$

where L_t^{-1} is the previously given integration operator. The solution $u(x,t)$ must satisfy the requirements imposed by the initial conditions. The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques.

ii) Homotopy Analysis Method:

In this section, the HAM [25, 26] is considered. The HAM has been constructed and successfully implemented to solve as an approximate and numerical solutions for many types of nonlinear problems [25-28] and the references cited therein. A very nice book for explaining the basic ideas of the HAM and its relationships with other analytic techniques, and some of its applications in science and engineering are given in Liao's book [25]. There are many same groups of methods with HAM in literature which are implemented nonlinear problems. Most of these groups of the methods are in principle based on Taylor series in an embedding parameter. If one could guess initial function and auxiliary linear operator well then one can get very good approximations in a few terms especially small value of the variable of the series.

For the purpose of illustration of the HAM [25], the fifth order KdV equation is written in the operator form as

$$L_t u + u u_x + u_{xxxxx} = 0, \tag{42}$$

where L is a linear operator: $L \equiv \frac{\partial}{\partial t}$. Eq. (42) can be written in a nonlinear operator form

$$N\left[\phi(x,t;q)\right] = \frac{\partial \phi(x,t;q)}{\partial t} + \\ +\phi(x,t;q)\frac{\partial \phi(x,t;q)}{\partial x} + \frac{\partial^5 \phi(x,t;q)}{\partial x^5}, \tag{43}$$

where $q \in [0,1]$ is an embedding parameter and $\phi(x,t;q)$ is a function.

From $u(x,0) = U_0(x)$, $-\infty < x < \infty$ it is straightforward to express the solution u by a set of base functions $\left\{e_n(x)t^n,\ n \ge 0\right\}$, where $e_n(x)$ as a coefficient is a function with respect to x. This provides us with the so-called Rule of Solution Expression.

Following Liao's method [25, 26], let $u(x,0) = U_0(x)$ indicate an initial guess of the exact solution u, $h \neq 0$ is an auxiliary parameter and $H(x,t) \neq 0$ is an auxiliary function. A zero-order deformation equation is constructed as $(1-q)L\left[\phi(x,t;q) - u_0(x,t)\right] = qhH(x,t)N\left[\phi(x,t;q)\right]$,

$$\tag{44}$$

with the initial condition

$$\phi(x,0;q) = U_0(x), \tag{45}$$

when $q = 0$ and 1 the above equation has the solution

$$\phi(x,t;0) = u_0(x,t), \tag{46}$$

and

$$\phi(x,t;1) = u(x,t), \tag{47}$$

respectively.

Assume, the auxiliary function $H(x,t)$ and the auxiliary parameter h are properly chosen so that $\phi(x,t;q)$ can be expressed by the Taylor series

$$\phi(x,t;q) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) q^n \tag{48}$$

where

$$u_n(x,t) = \frac{1}{n!}\frac{\partial^n \phi(x,t;q)}{\partial q^n}\bigg|_{q=0}, \tag{49}$$

and besides that the above series is convergent at $q = 1$. Using (46) and (47) then yield

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t). \tag{50}$$

For the sake of simplicity, define the vectors

$$\vec{u}_n(x,t) = \left\{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\right\}, \tag{51}$$

differentiating the zero-order deformation Eq. (44) n times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing by $n!$, the nth-order deformation equation is written as $L\left[u_n(x,t) - \chi_n u_{n-1}(x,t)\right] = hH(x,t) R_n\left[\vec{u}_{n-1}(x,t)\right]$

$$\tag{52}$$

where

$$R_n\left[\vec{u}_{n-1}(x,t)\right] = \frac{1}{(n-1)!}\left\{\frac{\partial^{n-1}}{\partial q^{n-1}} N\left[\sum_{m=0}^{\infty} u_m(x,t) q^m\right]\right\}\bigg|_{q=0}$$

or

$$R_n\left[u_{n-1}\right] = \frac{\partial u_{n-1}}{\partial t} + \sum_{i=0}^{n-1} u_i \frac{\partial u_{n-1-i}}{\partial x} + \frac{\partial^5 u_{n-1}}{\partial x^5}, \tag{53}$$

And

$$\chi_n = \begin{cases} 0, & n \le 1, \\ 1, & n > 1, \end{cases} \tag{54}$$

with the initial condition

$$u_n(x,0) = 0, \quad n \ge 1. \tag{55}$$

Therefore nth order approximation of $u(x,t)$ is given by

$$u(x,t) \approx u_0(x,t) + \sum_{m=1}^{N} u_m(x,t). \tag{56}$$

Substituting the fifth order KdV equation (35a) with initial value (35b) into Eq. (52) with (53) and using *Mathematica*, the series solutions of this equation can be constructed as an approximate series solution (56).

4. Numerical Experiments and Their Discussions

For numerical comparisons purposes, we consider fifth order KdV equation and different variation of the parameters of the considered solution (35b). The formula of numerical results for ADM and HAM (when $H\left(x,t\right)=1$) are given as follow

$$\lim_{n\to\infty}\varphi_n = u(x,t)$$

where

$$\varphi_n(x,t)=\sum_{k=0}^{n}u_k(x,t),\ n\geq 0. \qquad (57)$$

The recurrence relations of the methods are given as in (40) with (41), (52) with (53), respectively. Moreover, the decomposition series of the nonlinear equation's solutions are generally converging very rapidly in real physical problems [8, 28]. They obtained some results about the speed of convergence of this method providing us to solve linear and nonlinear functional equations. In the case of the convergence of the HAM is numerically shown in some work [29] and references there in.

In order to verify numerically whether the proposed methodologies lead to higher accuracy, the numerical solutions can be evaluated using the *n*-term approximation (57). These tabulated results are showing the difference of the exact solution and numerical solution of the absolute errors for the ADM and HAM.

Table 1. Comparison between the absolute errors of the solution (35b) of Eq. (35a) for $c_1=0.1,\ c_2=0.1,\ a_0=0.1,\ k=1$.

$\left|u_{Exact}-\varphi_5\right|$ for ADM

t | x	0.1	0.2	0.3	0.4	0.5
0.1	2.00373×10^{-12}	7.74349×10^{-11}	1.2881×10^{-9}	9.68869×10^{-9}	4.64916×10^{-8}
0.2	3.83693×10^{-13}	4.3201×10^{-11}	7.66676×10^{-10}	5.79455×10^{-9}	2.78193×10^{-8}
0.3	3.48166×10^{-13}	2.77467×10^{-11}	4.71651×10^{-10}	3.55213×10^{-9}	1.70397×10^{-8}
0.4	1.49214×10^{-13}	1.71951×10^{-11}	2.94827×10^{-10}	2.2216×10^{-9}	1.06554×10^{-8}
0.5	3.69482×10^{-13}	1.15037×10^{-11}	1.88827×10^{-10}	1.41763×10^{-9}	6.79402×10^{-9}

Table 2. Comparison between the absolute errors of the solution (35b) of Eq. (35a) for $c_1=0.1,\ c_2=0.1,\ a_0=0.1,\ k=1$.

$\left|u_{Exact}-\varphi_5\right|$ for HAM with value of $h=-1$

t | x	0.1	0.2	0.3	0.4	0.5
0.1	1.19299×10^{-8}	3.84519×10^{-7}	2.94122×10^{-6}	0.0000124852	0.0000383831
0.2	7.8463×10^{-9}	2.52815×10^{-7}	1.93315×10^{-6}	8.20325×10^{-6}	0.0000252105
0.3	5.25758×10^{-9}	1.69354×10^{-7}	1.29457×10^{-6}	5.49167×10^{-6}	0.0000168722
0.4	3.58359×10^{-9}	1.154×10^{-7}	8.81892×10^{-7}	3.74006×10^{-6}	0.0000114872
0.5	2.48115×10^{-9}	7.9878×10^{-8}	6.10274×10^{-7}	2.58747×10^{-6}	7.94504×10^{-6}

In Table 1-2, we have tabulated the numerical results of ADM and HAM. In our experience, HAM has advantage as well as ADM because if we could guess right value of auxiliary parameter *h* then numerical results are very accurate [25-28]. These correct choices of *h* value almost prove of the convergence of the HAM [30, 31]. Because of this fact, the considerations of convergence of the series solution depend on the right value of the auxiliary parameter *h* so that we have obtained and illustrated absolute errors of solutions of different values by using different values of *h* which are tabulated in Table 1 for the various values of h for interval $-1.4 \le h \le -0.6$ and similar constants values as in the Table 1 and Table 2-3. It is also our experience to choose of the h value depends on the taken equation and its solution, too [32-35].

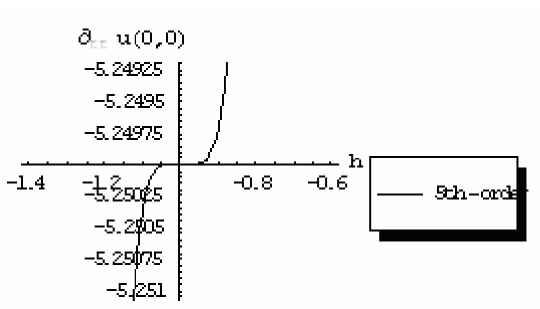

Fig. 1. The h-curve of $u_{tt}(0,0)$ at the 5th order of approximation when $H(x,t)=1$.

We also depicted h-curve at the 5th order of approximation and this test proof our test of the value $h=-1$ is correct which is shown in Fig.1. In both of these observations show that $h=-1$ is appropriate value to use and get the numerical results by using HAM. Moreover, when $h=-1$ and the approximate series (57) is as the same series solution obtained by ADM.

Table 3. Comparison between the absolute errors of the solution for Eq. (35a) with different values of auxiliary parameter h for HAM when $H(x,t)=1$.

t	HAM $(h=-1.4)$	HAM $(h=-1.2)$	HAM $(h=-1)$	HAM $(h=-0.8)$	HAM $(h=-0.6)$
0.1	1.40378×10^{-9}	1.80966×10^{-14}	0	3.27516×10^{-14}	1.87587×10^{-9}
0.2	2.26856×10^{-9}	2.44249×10^{-14}	2.22045×10^{-16}	8.11573×10^{-14}	4.09078×10^{-9}
0.3	2.71101×10^{-9}	2.33147×10^{-14}	1.11022×10^{-16}	1.49769×10^{-13}	6.67937×10^{-9}
0.4	2.82943×10^{-9}	1.78746×10^{-14}	1.11022×10^{-16}	2.4325×10^{-13}	9.67888×10^{-9}
0.5	2.70604×10^{-9}	9.99201×10^{-15}	4.44089×10^{-16}	3.67595×10^{-13}	1.31293×10^{-8}

This reality has discussed in the paper written by Sun et al. [36]. This is maybe the case of their considered problem but in our implementation there are some similarity of these two methods as numerically but one can see there are some differences of these two techniques. In our opinion, one clear difference of these methods is ADM is taking each terms of the series and equation terms ones and it does not need to use u_t terms of the Eq. (35a) in our case but HAM is using this term of the Eq. (35a) in every iterations it maybe makes difference for two methods. In our opinion, the approximate solution of HAM logically contains the approximate solution of ADM [25] but not exactly.

The numerical solutions a special value of $h=-1$ for the HAM perform very well one can see the Table 3. It is also valuable opportunity for us to use HAM among the considered methods because it gives the users a flexibility to choose different values of h which its value is necessary for our problem.

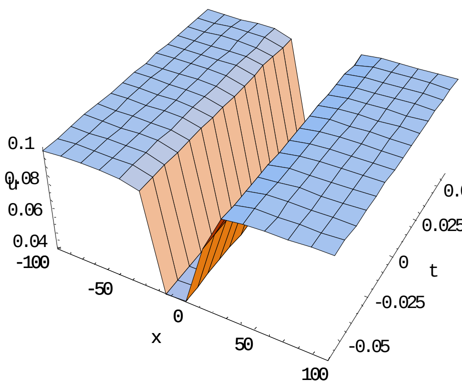

Fig. 2. A numerical illustration of exact solution (35a) at the different values of t and x for $c_1=0.1,\ c_2=0.1,\ a_0=0.1,\ k=1$.

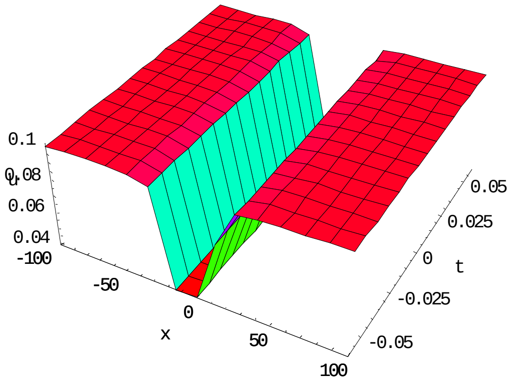

Fig. 3. A numerical illustration approximation solution of (35a) by the ADM at the different values of *t* and *x* for $c_1 = 0.1,\ c_2 = 0.1,\ a_0 = 0.1,\ k = 1$.

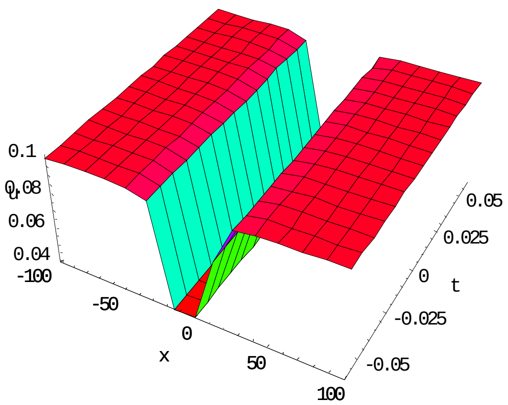

Fig. 4. A numerical illustration approximation solution of (35a) by the HAM ($h = -1$) at the different values of *t* and *x* for $c_1 = 0.1,\ c_2 = 0.1,\ a_0 = 0.1,\ k = 1$.

The graphs of the numerical values of the exact and fifth terms of the approximate series solutions by using ADM and HAM of the fifth order KdV equation are depicted in Fig.2, 3, 4. These four terms are very similar for this particular equation and its solutions. It is achieved a very good approximation with the actual solution of the equation by using five terms only of the series derived above (57) for all considered methods with the value of $c_1 = 0.1,\ c_2 = 0.1,\ a_0 = 0.1,\ k = 1$ which are depicted in Fig. 2,3,4. This is because of the nature of the series methods. In fact, these are illustrated in Tables 1, 2, 3. It is clear evident that the overall errors can be made smaller by adding new terms of the series (57) by using ADM and HAM. There are some zero values in tables which are not the really zero. These are produced by computer because of defining the digital.

5. Conclusion and Remarks

In this paper, we present a direct algebraic method by using Eq. (4) and with aid of *Mathematica*, implement it in a computer algebraic system. An implementation of the method is given by applying it fifth order KdV equation. The method can be used to many other nonlinear evolution equations or coupled ones. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation on a computer by the help of symbolic programs such as *Mathematica*, *Maple*, *Matlab*, and so on.

In last, a clear conclusion can be draw from the numerical results that the ADM algorithm provides highly accurate numerical solutions without spatial discretizations for nonlinear partial differential equations. It is also worth noting that the advantage of the approximation of the series methodologies displays a fast convergence of the solutions. The illustrations show the dependence of the rapid convergence depends on the character and behavior of the solutions just as in a closed form solutions. Finally, we point out that, for given equations with initial values $u(x,0)$, the corresponding analytical and numerical solutions are obtained according to the recurrence relations (41) and (52) using *Mathematica* package version of *Mathematica* 4 in PC computer.

Nonlinear phenomena play a crucial role in applied mathematics and physics. Furthermore, when an original nonlinear equation is directly calculated, the solution will preserve the actual physical characters of solutions. Explicit solutions to the nonlinear equations are of fundamental importance. Various effective methods have been developed to understand the mechanisms of these physical models, to help physicians and engineers and to ensure knowledge for physical problems and its applications.

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