Solving System of Linear Equations on the OTIS-Arrangement Network

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Abstract—In the last two decades a lot of research efforts have conducted and concluded that the Optical Transpose Interconnection Systems (OTIS) is one of the promising candidates for high speed parallel computers (HSPC) [15, 16, 17, 21]. In this paper, we continue what we have started in a previous research [26] and we apply solving system of linear equations on the grid structure for the OTIS-Arrangement Network (OTIS-AN). The proposed algorithms for solving system of linear equations for OTIS-AN could be utilized and customized for any other type of OTIS networks which will save efforts and time of other researchers. The proposed algorithms are based on grids structure as popular structure that support a vast body of parallel applications including linear algebra, divide-and-conquer type of algorithms, sorting, and FFT computation. This research confirms the viability of the OTIS-AN as an attractive alternative for HSPC.

Keywords—Parallel Computing, Arrangement Networks, Optoelectronic Systems, Interconnection Networks, Parallel Algorithms.

I. INTRODUCTION

One of the important issues that should be considered when designing a parallel system is the choice of the network topology that involves inherent trade-offs in terms of efficient algorithms support and network implementation cost. For instance, networks with large bisection width allow fast and reliable communication. However, such networks are difficult to implement using today’s electronic technologies that are two dimensional in nature [2, 4, 6, 7]. Free-space optical technologies offer several features to improve this trade-off. The improved transmission rates, dense interconnects, power consumption, and signal interference are few examples on these advantages [1, 10, 11, 12].

In this paper, we focus on the Optical Transpose Interconnection Systems-Arrangement Network which was proposed by Awwad and Sadi that can be easily implemented using free-space optoelectronic technologies [21]. In OTIS-AN processors are partitioned into groups, where each group is realized on a separate chip with electronic inter-processor connects. Processors on separate chips are interconnected through free space interconnects. The philosophy behind this separation is to utilize the features of both the optical and the electronic technologies.

The advantage of using OTIS-AN as optoelectronic architecture lies in its ability to trace the fact that free space optical communication is superior in terms of speed and power consumption when the connection distance is more than few millimeters [20]. In the OTIS-AN, shorter (intra-chip) communication is realized by electronic interconnects while longer (inter-chip) communication is realized by free space interconnects.

Extensive result efforts for the OTIS in general have been reported in [18, 19]. The achievable Terra bit throughput at a reasonable cost makes the OTIS-AN a strong competitive to its factor network arrangement [1, 6, 7, 11, 12]. These encouraging findings prompt the need for further testing of the suitability of the OTIS-AN for real-life applications. This research along with some previous research has been conducted in this direction [5, 8, 15, 17, 18, 26]. Valuable results have presented and evaluated various algorithms on OTIS-networks such as basic data rearrangements, routing, selection and sorting [13, 17], but still there is apart from the above mentioned works, solving of systems of linear equations on the OTIS is still yet to mature. In this research we contribute towards filling this gap by presenting algorithms for solving system of linear equations on the OTIS-AN.

The proposed algorithms for solving the system of linear equations is discussed in the sequel, before that we will present the needed definitions and notations of the topological properties of OTIS-AN. Furthermore the grid structure will be presented.

II. TOPOLOGICAL PROPERTIES OF OTIS-AN

This section reviews some of the basic topological properties of the OTIS-Arrangement network including size, degree, diameter, number of links, and shortest distance between two nodes [7, 21]. The topological properties of the OTIS-Arrangement network along with those of the Arrangement network are discussed below. Before disusing these properties let us introduce some basic definitions and terminologies for the arrangement network along with OTIS networks.

Definition 1: Let $n$ and $k$ be two integers satisfying $1 \leq k \leq n-1$ and let us denote $<n> = \{1, 2, \ldots, n\}$ and $<k> = \{1, 2, \ldots, k\}$. Let $P_n^k$ taken $k$ at a time, the set of arrangements of $k$
elements out of the \( n \) elements of \(<n>\). The \( k \) elements of an arrangements \( p \) are denoted \( p_1, p_2, \ldots, p_k \).

**Definition 2:** The \((n,k)\)-arrangement graph \( A_{n,k} = (V, E) \) is an undirected graph given by:

\[
V = \{ p_1, p_2, \ldots, p_k \mid p_i \in \langle n \rangle \text{ and } p_i \neq p_j \text{ for } i \neq j \} = P^n_k, \quad \ldots \quad (1)
\]

and

\[
E = \{(p, q) \mid p \text{ and } q \text{ in } V \text{ and for some } i \in <k>, p_i \neq q_i \text{ and } p_j = q_j \text{ for } j \neq i \} \ldots \quad (2)
\]

That is the nodes of \( A_{n,k} \) are the arrangements of \( k \) elements out of \( n \) elements of \( <n>\), and the edges of \( A_{n,k} \) connect arrangements which differ exactly in one of their \( k \) positions. For example in \( A_{5,2} \) the node \( p = 23 \) is connected to the nodes \( 21, 24, 25, 13, 43, \) and \( 53 \). An edge of \( A_{n,k} \) connecting two arrangements \( p \) and \( q \) which differ only in one position \( i \), it is called \( i \)-edge. In this case, \( p \) and \( q \) is called the \((i.q)\)-neighbour of \( p \). \( A_{n,k} \) is therefore a regular graph with degree \( k(n-k) \) and \( n!/(n-k)! \) nodes. As an example of this network figure 1 shows \( A_{4,2} \) arrangement with size of 12 nodes and a symmetric degree of 4.

Since OTIS-AN are basically constructed by "multiplying" a known topology by itself. The set of vertices is equal to the Cartesian product on the set of vertices in the factor network. The set of edges consists of edges from the factor network and new edges called the transpose edges. The formal definition of OTIS-networks is given below.

**Definition 3:** Let \( G_0 = (V_0, E_0) \) be an undirected graph representing a factor network. The OTIS-\( G_0 = (V, E) \) network is represented by an undirected graph obtained from \( G_0 \) as follows \( V = \{(x, y) \mid x, y \in V_0 \} \) and \( E = \{(x, y), (x, z) \mid \text{if } (y, z) \in E_0 \} \cup \{(x, y), (y, x) \mid x, y \in V_0 \text{ and } x \neq y \} \).

![Figure 1: The arrangement graph \( A_{4,2} \)](attachment:figure1.png)

The set of edges \( E \) in the above definition consists of two subsets, one is from \( G_0 \), called \( G_0 \)-type edges, and the other subset contains the transpose edges. The OTIS-AN approach suggests implementing Arrangement-type edges by electronic links since they involve intra-chip short links and implementing transpose edges by free space optics. Throughout this paper the terms “electronic move” and the “OTIS move” (or “optical move”) will be used to refer to data transmission based on electronic and optical technologies, respectively.

We will refer to \( g \) as the group address and \( p \) as the processor address. An intergroup edge of the form \( (\langle g, p \rangle, \langle p, g \rangle) \) represents an optical link and will be referred to as OTIS or optical move. Note that also we will be using the following notations:

- \( |A_{n,k}| = \text{size of the graph } A_{n,k} \).
- \( |\text{OTIS-}A_{n,k}| = \text{size of the graph OTIS-}A_{n,k} \).
- \( \text{Deg. } A_{n,k} (p) = \text{Degree of the graph } A_{n,k} \text{ at node } p \).
- \( \text{Deg. } \text{OTIS-}A_{n,k} (g, p) = \text{Degree of the graph OTIS-}A_{n,k} \text{ at node address } \langle g, p \rangle \).
- \( \text{Dist-}A_{n,k} (p_1, p_2) = \text{The length of a shortest path between the two nodes } p_1 \text{ and } p_2 \text{ in Arrangement graph} \).
- \( \text{Dist. } \text{OTIS-}A_{n,k} (p_1, p_2) = \text{The length of a shortest path between the two nodes } \langle g_1, p_1 \rangle \text{ and } \langle g_2, p_2 \rangle \text{ in OTIS-Arrangement} \).

In the OTIS-AN the notation \( \langle g, p \rangle \) is used to refer to the group and processor addresses respectively. Figure 2 shows that as an example of OTIS-\( A_{3,2} \). The figure shows that two nodes \( \langle g_1, p_1 \rangle \) and \( \langle g_2, p_2 \rangle \) are connected if and only if \( g_1 = g_2 \) and \( p_1 \neq p_2 \in E_0 \) (such that \( E_0 \) is the set of edges in Arrangement network) or \( g_1 = p_2 \) and \( p_1 = g_2 \) in this case the two nodes are connected by transpose edge. The distance in the OTIS-Arrangement is defined as the shortest path between any two processors, \( \langle g_1, p_1 \rangle \) and \( \langle g_2, p_2 \rangle \), and involves one of the following forms:

1. When \( g_1 = g_2 \) then the path involves only electronic moves from source node to destination node.
2. When \( g_1 \neq g_2 \) and if the number of optical moves is an even number of moves and more than two, then the paths can be compressed into a shorter path of the form: \( \langle g_1, p_1 \rangle \xrightarrow{E} \langle g_1, p_2 \rangle \xrightarrow{O} \langle g_2, p_2 \rangle \) here the symbols E and O stand for electronic and optical moves respectively.
3. When \( g_1 = g_2 \) and the path involves an odd number of OTIS moves. In this case the paths can be compressed into a shorter path of the form:

\[
\langle g_1, p_1 \rangle \xrightarrow{E} \langle g_2, p_1 \rangle \xrightarrow{O} \langle g_2, p_2 \rangle \xrightarrow{E} \langle g_2, p_2 \rangle.
\]

The following are the basic topological properties for the OTIS-Arrangement. For instance if the factor Arrangement network is of size \( n!/(n-k)! \), degree is \( n-1 \) and diameter is \( \lceil 1.5 k \rceil \) \([7, 21]\). Then the size, the degree, the diameter, number of
links, and the shortest distance of OTIS-Arrangement network are as follows:

- Size of $|OTIS\cdot A_{n,k}| = |n!/(n-k)!|^2$.
- Degree of $OTIS\cdot A_{n,k}=\text{Deg}(A_{n,k})$, if $g = p$.
- Degree of $OTIS\cdot A_{n,k}=\text{Deg}(A_{n,k}) + 1$, if $g \neq p$.
- Diameter of $OTIS\cdot A_{n,k} = 2\lfloor 1.5 \, k \rfloor + 1$.
- Number of Links: Let $N_0$ be the number of links in the $A_{n,k}$ and let $M$ be the number of nodes in the $A_{n,k}$. The number of links in the $OTIS\cdot A_{n,k} = (M^2 - M)/2 + N_0^2$. For instance, the number of links in the $OTIS\cdot A_{2,2}$ consisting of 144 processors is $(12^2 - 12)/2 + 230^2 = 595$.

$$\text{Dist}(p, p') = \begin{cases} \min(d(p, g_1) + d(g_1, p') + 1, d(p, p_1) + d(p_1, g_2) + 2) & \text{if } g_1 \neq g_2 \\ \text{Dist}(p, p') & \text{if } g_1 = g_2 \end{cases}$$

**Theorem 1:** The length of the shortest path between any two processors $\langle g_1, p_1 \rangle$ and $\langle g_2, p_2 \rangle$ in OTIS-AN is $d(p_1, p_2)$ when $g_1 = g_2$, and $\min\{d(p_1, p_2) + d(g_1, g_2) + 2, d(p_1, g_2) + d(g_1, p_2) + 1\}$ when $g_1 \neq g_2$, where $d(p, g)$ stands for the shortest distance between the two processors $p$ and $g$ using any of the possible shortest paths as seen in the above forms 1, 2 and 3 [8].

It is obvious from the above theorem that when $g_1 = g_2$, then the length of the path between the two processors $\langle g_1, p_1 \rangle$ and $\langle g_2, p_2 \rangle$ is $d(p_1, p_2)$. From the shortest path construction methods in (2) and (3) above, it can be easily verified that the length of the path equal $\min\{d(p_1, p_2) + d(g_1, g_2) + 2, d(p_1, g_2) + d(g_1, p_2) + 2\}$. The proof of the above theorem is a direct result from (3).

### Table 1: below summarizes the topological properties of OTIS-AN along with its factor $A_{n,k}$.

<table>
<thead>
<tr>
<th>Network</th>
<th>Size</th>
<th>Degree</th>
<th>Diameter</th>
<th>Number of links</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{n,k}$</td>
<td>$n!/(n-k)!$</td>
<td>$(n(n-k))$</td>
<td>$(\frac{3}{2}K)$</td>
<td>$(n!/(n-k)! # [1.5 , k]) / 2$</td>
</tr>
<tr>
<td>OTIS-AN</td>
<td>$</td>
<td>n!/(n-k)!</td>
<td>^2$</td>
<td>$\text{Deg}(A_{n,k})$</td>
</tr>
</tbody>
</table>

![Figure 2. OTIS-A3,2 network](image)

**III. GRID STRUCTURE FOR OTIS-AN**

In this section the hierarchical structure of the OTIS-AN is discussed. The properties of a new decomposition method for the OTIS-AN presented and proved. These properties are then used in the subsequent sections to develop grids as method for solving system of linear equations on the OTIS-AN [7, 9, 21, 26].

An OTIS-AN based computer contains $N^2$ processors partitioned into $N$ groups with $N$ processors each. A processor is indexed by a pair $(x, y)$, $0 \leq x, y < N$ where $x$ is the group index and $y$ is the processor index. Processors within a group are connected by a certain interconnecting topology; while inter-group links are achieved by transposing group and processor indexes [8, 10, 11].

Since the OTIS-AN constructed by "multiplying" the arrangement factor topology by itself. It included that the vertex set is equal to the Cartesian product on the vertex set in the arrangement network. The edge set consists of edges from the arrangement network and new edges called the transpose edges.
The address of a node $u = (x, y)$ from $V$ is composed of two components: the first, denoted by $\rho(u) = x$, designates the group address and the second, denoted by $\omega(u) = y$, designates the processor address within that group.

The network OTIS-AN can be decomposed into $|V_0|$ disjoint copies of $A_{n,k}$. This decomposition can be achieved by fixing the group address and varying the processor address. Another way of decomposing the OTIS-A network is by fixing the processors address and varying the group address. These two decomposition methods are given below.

**Definition 4:** Let $\mathcal{Y}_i$ for all $i \in V_0$, be the subgraph induced by the set of nodes from $V$ having the form $(i, x)$ \forall x \in V_0$.

**Definition 5:** Let $\Phi_j$, for all $j \in V_0$, be the subgraph induced by the set of nodes from $V$ having the form $(x, j)$ for all $x \in V_0$. Given a graph $G$, for simplicity we denote by $V_G$ the set of vertices, $E_G$ the set of edges, $d_G(u, v)$ the length of a shortest path connecting $u$ and $v$, and $\delta_G$ the diameter of $G$. Finally, any two graphs $G_1$ and $G_2$ are said to share perfect matching if there exists a bijective function between $V_{G_1}$ and $V_{G_2}$.

**Definition 6:** Let $G_{\mathcal{Y}'} = (V_{G_{\mathcal{Y}'}} \cup \Phi_{\mathcal{Y}'})$ be the graph obtained from OTIS-AN0 by clustering $\mathcal{Y}'$ into a single vertex labeled by $i$ and having a link between $i$ and $j$ if $\mathcal{Y}'_i$ and $\mathcal{Y}'_j$ share a perfect matching, i.e. $V_{G_{\mathcal{Y}'}} = V_0$ and $E_{G_{\mathcal{Y}'}} = \{(i, j) \mid \mathcal{Y}'_i \text{ perfectly matches } \mathcal{Y}'_j\}$.

**Theorem 2:** The two $\mathcal{Y}'$ and $\Phi$ decomposition methods of the OTIS-AN0 have the properties:

1. $\mathcal{Y}'$ is isomorphic to $A_{n,0}$.
2. $V_{\mathcal{Y}'} \cap V_{\Phi} = \{i, j\}$.
3. $\mathcal{Y}'$ and $\Phi$ share perfect matching for all $i$ values.
4. $\mathcal{Y}'$ and $\Phi$ share perfect matching for all $i$ and $j$ values and hence $G_{\mathcal{Y}'}$ is a complete graph. (Figure 3)

**Proof:**

Property 1 is a direct consequence of Definition 4. The function $\rho$ maps nodes from $V_{\mathcal{Y}'}$ to $V_0$. In fact, the set $\{\rho(u) \mid u \in V_{\mathcal{Y}'}\}$ is equal to $V_0$ for any $i$. Since any two neighboring nodes $u$ and $v$ in $V_{\mathcal{Y}'}$ should have $\rho(u) = \rho(v)$ and since $(\rho(u), \rho(v))$ is an edge in $E_0$, the subgraph $V_{\mathcal{Y}'}$ is isomorphic to $A_{n,0}$.

Property 2 states that for any two labels $i$ and $j$ from $V_0$, the two subgraphs $V_{\mathcal{Y}'}$ and $V_{\Phi}$ have exactly one node in common. Since, $V_{\mathcal{Y}'} = \{(i, x) \mid x \in V_0\}$ and $V_{\Phi} = \{(x, j) \mid x \in V_0\}$, the intersection $V_{\mathcal{Y}'} \cap V_{\Phi}$ contains only the node $(i, j)$.

Let $f_i : V_{\mathcal{Y}'} \rightarrow V_{\Phi}$ be a function that maps nodes form $V_{\mathcal{Y}'}$ into $\Phi_j$ for all $i$ values defined as follows: $f_i((x, y)) = (y, x)$. First we have $|V_{\mathcal{Y}'}| = |V_{\Phi}|$ for all $i$ and $j$. For any two distinct nodes $u$ and $v$ in $V_{\mathcal{Y}'}$, we have $f_i((\rho(u), \rho(v))) = (\rho(u), \rho(v)) \neq (\rho(v), \rho(u)) = (\rho(v), \rho(u))$; because $\rho(u) \neq \rho(v)$. Hence the function $f_i$ is on-to-one and onto.

Let $t_i : V_{\mathcal{Y}'} \rightarrow V_{\mathcal{Y}'}$ be a function that maps nodes form $V_{\mathcal{Y}'}$ into $\mathcal{Y}'_i$, for any $i$ and $j$, as follows: $t_i((i, x)) = (j, x)$. For any two distinct nodes $u$ and $v$ from $V_{\mathcal{Y}'}$ we have $t_i((\rho(u), \rho(v))) = (j, \rho(u)) \neq (i, \rho(v)) = (j, \rho(v))$. Since $|V_{\mathcal{Y}'}| = |V_{\mathcal{Y}'}|$ it follows that $\mathcal{Y}'_i$ and $\mathcal{Y}'_j$ share perfect matching for all $i$ and $j$ values and hence $G_{\mathcal{Y}'}$ is a complete graph.

**Lemma 1:** $G_{\mathcal{Y}'}$ can be embedded into OTIS-AN0 with dilation $\delta_{A_{n,0}} + 2$.

**Proof:** Since $G_{\mathcal{Y}'}$ is complete, any two distinct nodes $i$ and $j$ in $V_{\mathcal{Y}'}$ are neighbors. The “virtual” path between $(i, x)$ and $(j, x)$ in OTIS-AN0 that corresponds to the edge $(i, j)$ in $E_{AN0}$ is constructed as follows: $(i, x) \rightarrow (x, i) \parallel \pi_{G_0}(i, j) \parallel (x, j) \rightarrow (j, x)$. An arrow represents an edge connecting the two nodes and the operation “$|$” means appending two paths (i.e. connecting the last node in the left path to first node in the right path). Notice that the choice of $x$ from $V_0$ does not affect the construction of this path nor its length. The path segment $\pi_{G_0}(i, j)$ is an isomorphic copy to the optimal length path from $i$ to $j$ in $A_{n,0}$. It can be verified that the above constructed path is of optimal length equal to $d_{A_{n,0}}(i, j) + 2$. Hence, the longest such path cannot exceed $\delta_{A_{n,0}} + 2$.

**IV. EXPLOITING THE OTIS-AN GRID STRUCTURE FOR SOLVING SYSTEM OF LINEAR ALGEBRA**

In this section we apply the proposed grid structure for solving systems of linear equations, $A \overrightarrow{x} = \overrightarrow{b}$, a problem that arises in many areas of science and engineering [7, 8, 21]. A direct method of solution transforms the system $A \overrightarrow{x} = \overrightarrow{b}$ into $U \overrightarrow{x} = \overrightarrow{c}$ where $U$ is an upper triangular matrix. The solution vector $\overrightarrow{x}$ is then obtained by back substitution. The standard procedure to carry out matrix triangulation is Gaussian elimination (GE). The GE procedure generates a sequence of $\times n$ matrices $A^{(1)}, A^{(2)}, ..., A^{(n)}$ where $A^{(1)}$ is the initial matrix, $A$, and $A^{(n)}$ is the desired triangular matrix, $U$. The matrix $A^{(k)}$ ($k=2, ..., n$) represents the equivalent linear system for which the variable $x_k-1$ has just been eliminated.
In this implementation of matrix triangulization we use the standard 2-D matrix distribution method that is characterized by "combining" two functions; the first partitions the augmented matrix $A | \bar{b}$ into sets of elements and the second assigns these parts to the processors [9]. In the sequel we present a practical version of this distribution method.

In the proposed $|V_0| \times |V_0|$ grid structure, rows are the disjoint $\Psi_i$'s and columns are the disjoint $\Phi_i$'s. Following the notation of Theorem 2, the processor at the $r^{th}$ row and the $c^{th}$ column of the grid (denoted by $p_{rc}$) is laid down at the intersection of $\Psi_r$ and $\Phi_c$, where $r = h(x)$ and $c = h(y)$. Recall that $h$ is a function that ranks nodes of $V_0$.

For any positive integer $t$, we denote by $\langle t \rangle$ the set of integer numbers from 1 to $t$. Let $\Theta: \langle |V_0| \rangle \rightarrow 2^{|V_0|}$ be an injective function that maps processor row indexes into subsets of matrix row indexes. Similarly, let $\Lambda: \langle |V_0| \rangle \rightarrow 2^{|V_0|}$ be an injective function that maps processor column indexes into subsets of matrix column indexes. Our matrix distribution method is characterized by the function $\xi = \Theta \times \Lambda$. The function $\xi$ maps ordered pairs from $\langle |V_0| \rangle \times \langle |V_0| \rangle$ into sets of ordered pairs from $2^{|V_0|} \times 2^{|V_0|}$. The set $\xi(r, c)$ induces the set of elements from $A | \bar{b}$ to be assigned to $p_{rc}$.

The function $\xi$ covers the class of matrix distribution methods that map elements of the same row (resp. column) to a processor in $\Psi_i$ for some $x$ (resp. $\Phi_i$ for some $y$). These are called block-distribution methods. In the sequel, we consider only those instances of $\xi$ that are based on block-distribution and map the matrix evenly to the set of processors. Furthermore, we assume $\Theta(r_i) \cap \Theta(r_j) = \phi$ for $r_i \neq r_j$ and $\Lambda(c_1) \cap \Lambda(c_2) = \phi$ for $c_1 \neq c_2$. Figure 4 shows an example of block matrix distribution and the relationship between rows and columns in a 16-node OTIS-mesh.

The broadcast-based parallel GE algorithm for OTIS-$A_{n,k} 0$, abbreviated $b$ OTIS, reduces the matrix $A$ of order $n$ to a triangular form using $n - 1$ steps. At any step, $k$, the following tasks have to be performed by $p_{rc}$ in the order given below.

Task $\pi_p = \{ $ pivoting $ \} $ transform the system so that
\[ a_{ik} = \max \{ a_{ik} \} \text{ for } k \leq i < n \] - partial pivoting
\[ a_{ik} = \max \{ a_{ik} \} \text{ for } k < i, j \leq n \] - complete pivoting

Task $\pi_m = \{ $ compute multipliers $ \} a_{ik} = a_{ik} / a_{kk}$ for all $i \in \Theta(r)$ such that $k + 1 < i \leq n$

Task $\pi_x.c = \{ $ eliminate $ \} a_{ij} = a_{ij} - a_{ik} a_{kj}$ for all $i \in \Theta(r)$ and $j \in \Lambda(c)$ such that $k + 1 < i, j \leq n$

Let $\pi_y^M$ denote the task of broadcasting a pivot subrow of $M$ elements in $\Phi_i$, and let $\pi_y^M$ denote the task of broadcasting a multiplier subcolumn of $M$ elements in $\Psi_i$. Notice that broadcasting in $\Phi_i$ uses the usual communication algorithms of $G_0$ except that two data shift operations have to "bracket" communication algorithms to transfer data from $\Phi_i$ to $\Psi_i$ and back by using the disjoint transpose edges (Property 2 in Theorem 2).

The algorithm $b$ OTIS executed by the node $p_{rc}$ in the $|V_0| \times |V_0|$ grid is shown in Figure 5. The steps of $b$ OTIS can be summarized as follows: If $p_{rc}$ holds a pivot, it performs $\pi_p$ followed by $\pi_{y^{(n+1)/|V_0|}}$, otherwise it waits until it receives a pivot. Next, if $p_{rc}$ holds a multiplier, it performs $\pi_{x^c}$ and then $\pi_{y^{n/|V_0|}}$, otherwise it waits until it receives a multiplier. Finally, $p_{rc}$ performs $\pi_x^{c.r}$ and enters the next iteration.

For partial pivoting, the $k^{th}$ pivot row is determined by processors in $\Phi_{k+1}$ where $k \in \Lambda(c)$. These processors perform an "exchange-max" operation. At the end of this operation each processor will have a copy of the index of the pivot row, say $l$. If both $l$ and $k$ are in $\Theta(r)$ then the two subrows $k$ and $l$ are locally swapped; otherwise they are interchanged between all pairs of corresponding processors in $\Psi_{k+1}$ and $\Psi_{l+1}$ for $l \in \Theta(v)$. The algorithm for the task $\pi_p$ with partial pivoting is given in Figure 6.
To carry our complete pivoting all the processors in the grid perform exchange-max operation to find \( l \) and \( m \) such that \( |a_{l,m}| = \max \{|a_{i,j}| \text{ for } i \leq l, j \leq n\} \). Then, the \( k^{\text{th}} \) pivot row is located by swapping/interchanging the rows \( k \) and \( l \) and swapping/interchanging the columns \( k \) and \( m \). A complete pivoting algorithm for \( b_{-}\text{OTIS} \) is given in Figure 7.

The algorithm \( b_{-}\text{OTIS} \) transforms the linear system \( A\tilde{x} = \tilde{b} \) into an upper triangular system \( U\tilde{x} = \tilde{c} \). Next we develop a broadcast-based backward substitution algorithm, abbreviated \( bs_{-}\text{OTIS} \), to obtain the final solution. In \( bs_{-}\text{OTIS} \) we assume that elements of the reduced upper triangular matrix \( U \) and \( \tilde{c} \) are still in their respective processors. A processor \( p_{v,c} \) in the algorithm performs \( n \) steps. A step, \( k \), consists of the following simple tasks: (1) solve and broadcast \( x_{k} \) and (2) update and broadcast \( \tilde{c}^{k} \). To simplify the algorithm design we introduce the following two new tasks:

```
Task \( \pi_{x}^{k} \) = \{ solve \} \ x_{k} = c_{k} / u_{kk}

Task \( \pi_{c}^{k} \) = \{ update \} \ c_{i} = c_{i} - u_{ik} x_{k} \text{ for all } i \leq k
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At the beginning of the backward substitution algorithm, a processor \( p_{v,c} \) in \( \Phi_{k-1}(c) \) that holds the set of elements \( \{ c_{i} \} \text{ for } i \leq (r) \) from \( \tilde{c} \) (denote this set by \( \bar{c} \)) should broadcast these elements in \( \Psi_{k-1}(c) \). Notice that initially the known vector \( \tilde{c} \) is stored in \( \Phi_{k-1}(c) \) where \( n+1 \leq j \leq n \).

The backward substitution algorithm then proceeds by performing the \( n \) substitution steps. At the \( k^{\text{th}} \) step, the processor \( p_{v,c} \) holding \( u_{kk} \) performs \( \pi_{x} \) and then broadcast \( x_{k} \) in \( \Phi_{k-1}(c) \).

All other peers of \( p_{v,c} \) in \( \Phi_{k-1}(c) \) update their portion of \( \tilde{c} \) vector and broadcast it in their respective \( \Psi_{k-1}(c) \)'s. Figure 8 outlines this algorithm.

```
Algorithm \( bs_{-}\text{OTIS} \)  
\{ executed by the node \( p_{v,c} \) in the \( [V_{s}] \times [V_{c}] \) grid \}
for \( n+1 \leq j \leq N \)
\( |a_{l,m}| = \max \{|a_{i,j}| \text{ for } i \leq l, j \leq n\} \) such that \( k \leq l, j \leq N \}
\{ execute \( \pi_{x} \) \}
\| perform exchange-max operation to exchange \( a_{l,m} \) among all processors in \( \Phi_{k-1}(c) \)
\| broadcast \( l \) in \( \Psi_{k-1}(c) \)
\| else participate in the broadcast initiated in \( \Psi_{k-1}(c) \)
\| \| if \( l \leq (r) \) and \( k \leq (r) \) swap subrows \( k \) and \( l \)
\| \| else interchange subrows \( k \) and \( l \) between \( \Psi_{k-1}(c) \) and \( \Psi_{k-1}(c) \)
end for
end \( bs_{-}\text{OTIS} \)
```

Figure 8: The algorithm \( bs_{-}\text{OTIS} \).
CONCLUSION

The study of algorithms on the Optical Transpose Interconnection Systems in general and for OTIS-AN as particular case is still far from being matured. In this paper, we have contributed towards filling this gap by solving system of linear equations for OTIS-AN based on grid structure as popular framework for HSPC for the OTIS-AN. The proposed algorithms for solving the system of linear equations could be utilized for any factor OTIS network which will save time and efforts of researchers.

Several topological properties including size, degree, and diameter, number of links, optimal path and shortest distance between any two nodes for OTIS-AN have been presented. The proposed OTIS-AN is an attractive alternative of its factor network in terms of its grid structure for solving system of linear equations by utilizing both electronic and optical technologies. As a future research work we could solve many other real life problems on OTIS-AN network including divide-and-conquer type of algorithms and Fast Fourier transforms.

REFERENCES