Analysis of One-Dimensional Response of an Elastic Body Under Dynamic Loads

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Abstract—Impulsive loads and step loads are examples of dynamic loads which produce transient waves. Analytical and numerical solutions to transient wave problems in elastic materials can be obtained by integral transform methods, particularly Fourier and Laplace transforms. The transient response of an axially-loaded bar was studied by performing the formulation in the Laplace transform space. The solution in the time domain was obtained by an appropriate numerical inverse Laplace transformation method. The results are compared with ANSYS, and the efficiency of the Laplace transform method is demonstrated.

Keywords—Dynamic loads, Laplace transform, Wave propagation

I. INTRODUCTION

The study of the response of elastic bodies under dynamic loads essentially boils down to wave propagation analysis. The subject matter is extremely crucial in designing durable structural components. The one-dimensional problems to be tackled in the present work will involve axially-loaded bars. The idea of studying the behavior of bars dynamically-loaded in the axial direction stemmed from the biomedical engineering. From a biomedical engineering point of view, an axially-loaded fixed-free bar under pulsating dynamic pressure well represents an implant.

The analysis of elastic wave propagation is well documented in the literature [1-2]. The present study introduces the one-dimensional wave equation. It will be shown that one-dimensional motions of a linear elastic material are governed by the one-dimensional wave equation.

Most of the waves encountered in real life applications, both natural and man-made, are transient. Transient waves are those which change with time; but they are not steady-state waves. Analytical and numerical solutions to transient wave problems in elastic materials can be obtained by integral transform methods, particularly Fourier and Laplace transforms. Most of the works obtained through a literature survey have been about the Laplace transform and Fourier transform methods applied to dynamic problems [3-6]. The transient response will be studied by performing the formulation in the Laplace transform space. The solution in the time domain will be obtained by an appropriate numerical inverse Laplace transformation method. The Laplace transform is preferred because it does not require damping in the response which is the case in perfectly elastic materials.

The results obtained using various analyses will be compared with each other to assess the accuracy and efficiency of the Laplace transform method.

II. THEORY

The longitudinal motion of a uniform rod is governed by the differential equation

\[
\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \quad 0 < x < L, \quad t > 0
\]

where \( c^2 = \frac{E}{\rho} \) and \( E \) is Young’s modulus.

A fixed-free rod with the axial force applied at the free end will be considered. Initial and boundary conditions accompanying the governing differential equation (1) are

\[
u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0,
\]

\[
u(0,t) = 0, \quad \frac{\partial u(L,t)}{\partial x} = \frac{P(t)}{EA}
\]

Also axial end forces to be considered are: Step force, Ramp or Linearly increasing force, Linearly decreasing force, Half-cycle sine pulse force, Step force with finite rise time and Triangular force.

A. Solutions for the Step Force with Finite Rise Time

A detailed analysis for the load type “Step force with finite rise time” will be presented, and subsequently, the outline of analyses and results for other cases will be listed.

Taking the Laplace transform of Eqs. (1-2) yields

\[
U''(x,s) - \frac{s^2}{c^2} U(x,s) = 0
\]

\[
U(0,s) = 0, \quad \frac{\partial U(L,s)}{\partial x} = L\left\{ \frac{P(t)}{EA} \right\} = \frac{1}{EA} L\{P(t)\}
\]

where \( U(x,s) = L\{u(x,t)\}, s \) being the complex Laplace parameter. Solving the differential equation (3), we find

\[
U(x,s) = c_1 e^{-sx} + c_2 e^{sx}
\]
From the first condition in Eq. (4), \( c_2 = -c_1 \) and so

\[
U(x,s) = c_1 \left( e^{\frac{s}{c} - e^{-\frac{s}{c}}} \right)
\]

(6)

From the hyperbolic equalities, the Equation (6) becomes

\[
U(x,s) = c_3 \sinh \frac{s}{c} x
\]

(7)

From the second condition in Eq. (4), we have

\[
c_3 \frac{s}{c} \cosh \frac{s}{c} L = \frac{1}{EA} L\{P(t)\}
\]

(8)

Using the Equation (8), \( c_3 \) can be found for different loading types. The forces which we applied to the bar must be firstly determined. This force will be dynamic and the following force as shown in Figure-1 will be used.

\[
P(t) = \begin{cases} 
P_0 \frac{t}{b}, & 0 < t < b \\ 
P_0, & b < t < 2b \end{cases}
\]

The load \( P(t) \) is a periodic function. so; the following equation is used for finding \( L\{P(t)\} \).

\[
L\{P(t)\} = \frac{P_0}{1 - e^{-2bs}} \left( \int_0^b P_0 \frac{t}{b} e^{-st} dt + \int_b^{2b} e^{-st} dt \right)
\]

(9)

The Equation (9) can be derived as:

\[
L\{P(t)\} = \frac{e^{bs} - e^{2bs} + bs}{bs^2 - bs^2 e^{2bs}}
\]

(10)

Then put it into Equation (8)

\[
c_3 = \frac{P_0 c}{EA} \frac{e^{bs} - e^{2bs} + bs}{bs^2 - bs^2 e^{2bs}}
\]

(11)

Substituting Equation (11) into the Equation (7) yields

\[
U(L,s) = \frac{P_0 c}{EA} \frac{Tanh \frac{s}{c} L - e^{bs} - e^{2bs} + bs}{bs^2 - bs^2 e^{2bs}}
\]

(12)

Equation (12) is the end-point displacement in Laplace space. Later, we use direct Laplace transform methods for finding displacement of rod under dynamic loading in real time.

Table 1: Axial Displacement Expressions of the End Point in Laplace Space.

<table>
<thead>
<tr>
<th>Load Types</th>
<th>Laplace Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Step Force</td>
<td>( \frac{P_0 c}{EA} \frac{Tanh \frac{s}{c} L}{s^2} )</td>
</tr>
<tr>
<td>2 Linearly Inc. Force</td>
<td>( \frac{P_0 c}{EA} \frac{Tanh \frac{s}{c} L}{bs^3} )</td>
</tr>
<tr>
<td>3 Linearly Dec. Force</td>
<td>( \frac{P_0 c (bs - 1)}{EA} \frac{Tanh \frac{s}{c} L}{bs^3} )</td>
</tr>
<tr>
<td>4 Half Cycle Sine Force</td>
<td>( \frac{P_0 c kTanh \frac{s}{c} L}{EA} \frac{ks^2 + k^2}{s (s^2 + k^2)} )</td>
</tr>
<tr>
<td>5 Step Force with Finite Rise Time</td>
<td>( \frac{P_0 c}{EA} \frac{Tanh \frac{s}{c} L - \frac{e^{bs} - e^{2bs} + bs}{s(b^2 - b^2 e^{2bs})}}{2} )</td>
</tr>
<tr>
<td>6 Triangular Force</td>
<td>( \frac{P_0 c}{EA} \frac{Tanh \frac{s}{c} L \frac{Tanh \frac{bs}{c}}{bs^3}}{2} )</td>
</tr>
</tbody>
</table>

B. The Durbin’s Inverse Laplace Transform

A numerical inverse Laplace transform technique is necessary to obtain the values in the time domain (Calm, 2009). For this purpose, Durbin’s inverse Laplace transform technique based on the fast Fourier transform is used (Durbin, 1974). Durbin’s formulation for inverse Laplace transform is summarized as follows:
\[ f(t_j) = \frac{2e^{\gamma_\Delta N}}{T} \left[ -\frac{1}{2} \text{Re}[F(\alpha)] + \text{Re} \left\{ \sum_{k=0}^{\infty} (A(k) + iB(k)) e^{i \left( \frac{2\pi k}{N} \right) j} \right\} \right] \]

(j = 0, 1, 2, ......... N - 1)  \hspace{1cm} (13)

where

\[ A(k) = \sum_{l=0}^{L} \text{Re} \left\{ F(a+i(k+1)N) \frac{2\pi}{T} \right\} \]

\[ B(k) = \sum_{l=0}^{L} \text{Im} \left\{ F(a+i(k+1)N) \frac{2\pi}{T} \right\} \]  \hspace{1cm} (14)

Here, \( i \) is the complex number, \( s_k = a + i \frac{2\pi k}{T} \) is the \( k \)th Laplace transform parameter. There are \( N \) units of equal time intervals and \( T \) is the solution interval. \( f(t) \) is calculated for all \( t_j = j \Delta t = jT/N, (j = 0, 1, 2, ... N - 1) \). The most suitable way for transform is using the \( 5 \leq aT \leq 10 \) interval. For numerical examples in this study, the value of \( aT \) is chosen as 6. Finally, results can be modified multiplying each term by Lanczos (Lk) factors to obtain better results in the Laplace domain (Narayanan, 1979).

\[
L_k \Rightarrow \begin{cases} 
L_0 = 1 \text{ for } k = 0 \\
L_k = \frac{\sin \frac{k\pi}{N}}{\frac{k\pi}{N}} \text{ for } k > 0 
\end{cases} \hspace{1cm} (15)
\]

III. NUMERICAL EXAMPLES

In this section, for given material properties, the end-point displacements will be calculated using the various methods outlined in previous sections and the results will be compared to assure the efficiency of the Laplace transform method as it is applied to the particular problem on hand. Two sets of material properties are used:

1) \( E = 2.6 \times 10^7 \text{ N/m}^2, \rho = 7920 \text{ kg/m}^3 \) (concrete rod),
2) \( E = 20.3 \times 10^7 \text{ N/m}^2, \rho = 7860 \text{ kg/m}^3 \) (steel rod).

The geometrical properties are \( L = 12 \text{ m}, A = 3 \times 10^2 \text{ m}^2 \). Dynamic loads applied all have the same magnitude of \( P_0 = 1 \text{ N} \).

The results are presented graphically using the following analyses for each loading type:

A1 (DURBIN): Inverse Laplace transformation of the end-point displacement expressions given in Table-1 is obtained. For this purpose, a computer program is coded in Mathematica to analyze dynamic response of rods.

A2 (ANSYS): Point by point magnitudes of the dynamic force are input to ANSYS and the end-point displacements are calculated.

These results are presented graphically in Figures 1-6 for both materials. The legend \( \text{AiMj} \) refers to analysis \( \text{Ai} \) and material \( \text{Mj} \).

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IV. CONCLUSION

In this study, the methods of problem formulation and solution of an axially-loaded bar modeled as a continuous system were analyzed. The motions of continuous system were obtained from Laplace transform method.

The transient response of an axially-loaded bar was studied by performing the formulation in Laplace space the solution in the time domain was obtained by an appropriate numerical inverse Laplace transformation method.

The results are compared with ANSYS, and the efficiency of Laplace transform method is demonstrated. For this purpose, MATHEMATICA programs were developed for solving the end point displacement equations of the bar. It was seen that the numerical results are similar. At the result, we can say that:

The Laplace transform method of solving the differential equation provides a complete solution, yielding both transient and forced vibration.

The Laplace transform method gives the simple way for solving present problem.

REFERENCES