Stability Analysis of Carbon Nanotubes (CNTs) Based on Modified Couple Stress Theory

B. Akgöz and Ö. Civalek

1University of Akdeniz, TURKEY, bekirakgoz@akdeniz.edu.tr
2University of Akdeniz, TURKEY, civalek@yahoo.com

Abstract—This paper is related to stability analysis of Carbon Nanotubes (CNTs) based on couple stress elasticity and Bernoulli-Euler beam theory. In the former approach, the scale effect is taken into consideration. The governing equation is obtained via variational approach and solved analytically. The scale effects are investigated and results are presented.

Keywords—Couple stress elasticity, buckling, size effect, Carbon nanotubes, Bernoulli-Euler beam.

I. INTRODUCTION

Carbon nanotubes (CNTs) are discovered by Iijima in 1991 [1]. CNTs have many good and special properties including chemical, electrical, mechanical, and thermal. Because of that, many researchers at around the world have been interested to mechanical modeling of CNTs [2-4]. So, extensive theoretical and experimental studies on mechanical properties of CNTs have been performed. In the past decade, bending, buckling and vibration behaviors of CNTs have been a subject of interest using atomistic, molecular dynamics or continuum model [5-9]. Moreover, many authors have employed a continuum or structural mechanics approach because of more practical and efficient modeling. It is also accepted that, the classical elasticity theory is not suitable for modeling of nanostructures. In order to consider the size effect for these systems, nonlocal continuum theory has generally used. For this purpose, beam and shell theories have been used by researchers [10-16].

It shown from the experimental study that, the size effect can not be interpreted implicitly by beam models based on Cauchy elasticity theory due to lack of material length scale parameters. Then, higher order continuum (nonlocal) theories, which contain additional material length scale parameters besides the classical material constants have been proposed to predict the size dependence of these nano- and micro-sized structures. Mostly generally known higher order theories are the Cosserat (micropolar) elasticity [17], Eringen’s nonlocal theory [18], strain gradient elasticity [19, 20] and couple stress theories [21-24]. Recently, a modified couple stress theory was proposed by Yang et al. [25] which contain only one additional material length scale parameter in addition to the classical material constants. Also, the couple stress tensor is symmetric in this theory.

Recently, this theory has been used for static and dynamic analyses of micro-sized structures [26-31].

In this study, the governing equation of CNTs in buckling is derived with the aid of variational principle and Bernoulli-Euler beam model on the basis of modified couple stress theory. CNTs are considered as simply supported beam.

II. FORMULATION

The modified couple stress theory was proposed by Yang et al. which contain only one additional material length scale parameter in addition to the Lame constants. In this theory, the strain energy density is the function of both the strain tensor and curvature tensor. Furthermore, the couple stress tensor is symmetric. The strain energy \( U \) in a linear elastic isotropic material occupying region \( \Omega \) based on the modified couple stress theory can be written as [25]

\[
U = \frac{1}{2} \iiint_{\Omega} \left( \sigma_{ij} \varepsilon_{ij} + m_{ij} \chi_{ij} \right) dv
\]  

where \( \sigma_{ij} \) is stress tensor, \( \varepsilon_{ij} \) is strain tensor, \( m_{ij} \) is daviatoric part of couple stress tensor, and \( \chi_{ij} \) is symmetric curvature tensor.

\[
\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G \varepsilon_{ij}
\]

\[
\varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right)
\]

\[
m_{ij} = 2l_2^2 G \chi_{ij}
\]

\[
\chi_{ij} = \frac{1}{2} \left( \theta_{i,j} + \theta_{j,i} \right)
\]

where \( \lambda \) and \( G \) are Lame constants, \( l_2 \) is a material length scale parameter, \( u \) is displacement vector, and \( \theta \) is rotation vector as

\[
\theta_i = \frac{1}{2} c_{ik} u_{k,j}
\]
The displacement components for Bernoulli-Euler beam model are given by

\[ u = -z\psi(x), \quad v = 0, \quad w = w(x) \]  

(7)

where \( u, v, w \) are \( x-, y-, z- \) components of displacement vectors, and \( \psi \) is the rotation of line elements along the central axis of beam given approximately by

\[ \psi \approx \frac{d w(x)}{d x} \]  

(8)

Using above equations in Equation (1) with some mathematical manipulations, the strain energy \( U \) is rewritten as [26]

\[ U = \frac{1}{2} \int_0^l \left( EI + G A l^2 \right) \left( w'^2 \right) dx \]  

(9)

Also, when we consider in addition the effect of the axial compressive force \( N \) in above equation, one obtains the following expression for the above strain energy \( U \)

\[ U = \frac{1}{2} \int_0^l \left( EI + G A l^2 \right) \left( w'^2 \right) dx - \frac{1}{2} \int_0^l N \left( w' \right)^2 dx \]  

(10)

The variation of strain energy [32]

\[ \delta U = \left[ \int_0^l \left( EI + G A l^2 \right) w'^4 + N w'^2 \right] \delta w dx \]  

\[ - \left[ \int_0^l \left( EI + G A l^2 \right) w'^2 + N w' \right] \delta w \bigg|_0^l + \left[ \int_0^l \left( EI + G A l^2 \right) w'^2 \right] \delta w \bigg|_0^l \]  

(11)

Furthermore, the variation of the boundary shear force \( V \) and boundary moment \( M \) reads as

\[ \delta W = - \left[ \left( -V \right) \delta w \bigg|_0^l + (M) \delta w \bigg|_0^l \right] \]  

(12)

According to the variational principle, the following equation can be written as

\[ \delta \left( U - W \right) = \left[ \int_0^l \left( EI + G A l^2 \right) w'^4 + N w'^2 \right] \delta w dx \]  

\[ - \left[ \int_0^l \left( EI + G A l^2 \right) w'^2 + N w' \right] \delta w \bigg|_0^l + \left[ \int_0^l \left( EI + G A l^2 \right) w'^2 \right] \delta w \bigg|_0^l = 0 \]  

(13)

It can be seen clearly that all terms must be equal to zero in the above variational equation. So, the governing equation of CNTs for buckling is obtained as

\[ \left( EI + G A l^2 \right) w'^4 + N w'^2 = 0 \]  

(14)

Also, the boundary conditions at \( x = 0, L \), reads as

\[ V = - \left( EI + G A l^2 \right) w'^2 - N w' \quad \text{or} \quad \delta w = 0 \]  

III. SOLUTION OF BUCKLING PROBLEM

The buckling equation in Eq. (14) for the CNTs using Bernoulli-Euler beam model can be written as [33]

\[ w(x) = c_1 \sin \lambda x + c_2 \cos \lambda x + \frac{1}{\lambda^2} (Ax + B) \]  

(16)

\[ \lambda^2 = \frac{N}{EI + G A l^2} \]  

(17)

where \( c_1, c_2, A, B \) are integration constants.

The boundary conditions for simply supported CNTs at the both ends are

\[ w = 0 \quad M_x + Y_{sy} = 0 \]  

(18)

where \( M_x \) and \( Y_{sy} \) are resultant moment and couple moment, respectively.

By using above boundary conditions in Eq.(16), the integration constants are obtained as

\[ A = B = 0, \quad c_1 \sin \lambda L = 0, \quad c_2 = 0 \]  

(19)

For a non-trivial solution, the following equation must be implemented.

\[ \sin \lambda L = 0, \quad \lambda = \frac{n \pi}{L} \]  

(20)

and buckling loads are given by

\[ N = \left( \frac{n \pi}{L} \right)^2 \left( EI + G A l^2 \right) \]  

(21)

as buckling loads of CNTs for simply supported case.

IV. NUMERICAL EXAMPLES

The results of buckling analysis of carbon nanotubes (CNTs) based on modified couple stress theory are presented in this section. The effects of the scale parameter on buckling of CNTs are discussed briefly. In the figures, the case of \( l = 0 \) represents the classical results. For illustration purpose, the CNTs have the following material properties: \( E = 1000 \ GPa, \ G = 420 \ GPa, \ d = 1 \ nm, \ A = 0.785 \ nm^2, \ I = 0.0491 \ nm^4 \) [13]. Figure 1 illustrates values of the critical buckling loads for various values of additional material length
scale parameter. It can be seen clearly that when additional material length scale parameter $l_2$ increases, the critical buckling load values increase, too. In contrary, when the ratio of length to diameter increases, the critical buckling load values decrease. Furthermore, Figure 2 shows both the variation of buckling loads for different modes and the effect of additional material length scale parameter on buckling load values. Similarly, mode number increase, values of buckling load increase rapidly. Especially, the effect of additional material length scale parameter is very important for higher modes.

![Figure 1: Values of the critical buckling loads $N_{cr} (nN)$ for various values of additional material length scale parameter](image1)

![Figure 2: The buckling loads $N (nN)$ for first five modes and different values of additional material length scale parameter $l_2$](image2)

**ACKNOWLEDGMENT**

The financial support of the Scientific Research Projects Unit of Akdeniz University is gratefully acknowledged.
Stability Analysis of Carbon Nanotubes (CNTs) Based on Modified Couple Stress Theory


